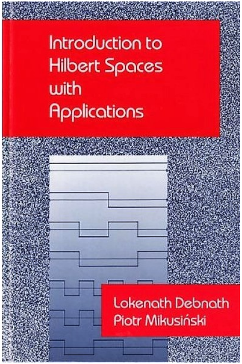
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.4. Norm in an Inner Product Space—Proofs of Theorems



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Theorem 3.4.1. Schwarz's Inequality

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For any x and y in an inner product space $|(x,y)| \leq \|x\| \, \|y\|$. Equality holds if and only if x and y are linearly dependent.

Proof. If $y = 0$, the inequality is satisfied. So without loss of generality, suppose $y \neq 0$. We know

$$0 \leq (x + \alpha y, x + \alpha y) = (x,x) + \overline{\alpha}(x,y) + \alpha(y,x) + |\alpha|^2(y,y).$$

Let $\alpha = -(x,y)/(y,y)$.

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Let $\alpha = -(x,y)/(y,y)$. Then

$$0 \leq (x,x) - \overline{\left(\frac{(x,y)}{(y,y)}\right)}(x,y) - \frac{(x,y)}{(y,y)}(y,x) + \left|\frac{(x,y)}{(y,y)}\right|^2 (y,y)$$

$$= (x,x) - \frac{|(x,y)|^2}{(y,y)} - \frac{|(x,y)|^2}{(y,y)} + \frac{|(x,y)|^2}{(y,y)} = (x,x) - \frac{|(x,y)|^2}{(y,y)}$$

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Proof (continued). Next, if x and y are linearly dependent then $y = \alpha x$ and the inequality reduces to equality. Conversely, suppose $|(x,y)| = \|x\| \, \|y\|$ or equivalently

$$(x,y)(y,x) = (x,x)(y,y). \qquad (*)$$

Then by $(*)$,

$$((y,y)x - (x,y)y, (y,y)x - (x,y)y)$$

$$= (y,y)^2(x,x) - (y,y)(y,x)(x,y) - (x,y)(y,y)(y,x) + (x,y)(y,x)(y,y) = 0.$$

Therefore $(y,y)x - (x,y)y = 0$ and x and y are linearly dependent. □

Corollary 3.4.1

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$$\|x+y\| \leq \|x\| + \|y\|.$$

Proof. We have

$$\begin{aligned}
\|x+y\|^2 &= (x+y, x+y) = (x,x) + (x,y) + \overline{(x,y)} + (y,y) \\
&= (x,x) + 2\mathrm{Re}((x,y)) + (y,y) \\
&\leq (x,x) + 2|(x,y)| + (y,y) \\
&\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \text{ by Schwarz's Inequality} \\
&\qquad\qquad\qquad\qquad\qquad \text{(Theorem 3.4.1)} \\
&= (\|x\| + \|y\|)^2.
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For any two elements x and y of an inner product space

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

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$$= \|x\|^2 + (x,y) + (y,x) + \|y\|^2. \qquad (*)$$

Replacing y with $-y$:

$$\|x-y\|^2 = \|x\|^2 - (x,y) - (y,x) + \|y\|^2. \qquad (**)$$

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