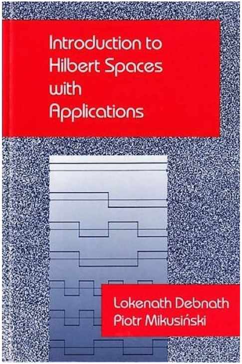
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.5. Hilbert Spaces—Definition and Examples—Proofs of Theorems



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Theorem/Example 3.5.6

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Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^2([a,b])$. Then

$$\int_a^b |f_m - f_n|^2 \to 0 \text{ as } m, n \to \infty.$$

By Schwarz's Inequality (Theorem 3.4.1),

$$|(|f_m - f_n|, 1)| = \left| \int_a^b |f_m - f_n| \right| = \int_a^b |f_m - f_n|$$

$$\leq \sqrt{\int_a^b 1}\sqrt{\int_a^b |f_m - f_n|^2} = \sqrt{b-a}\sqrt{\int_a^b |f_m - f_n|^2} \to 0$$

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as $m,n \to \infty$.

Theorem/Example 3.5.6 (continued)

Proof (continued). So $\{f_n\}$ is a Cauchy sequence in $L^1([a,b])$ and so converges in $L^1([a,b])$ (since it's a Banach space) to say $f \in L^1([a,b])$. Then $\int_a^b |f - f_n| \to 0$ as $n \to \infty$. By Theorem 2.8.2 (see page 58) there is a subsequence $\{f_{p_n}\}$ convergent to f a.e. We can choose m, n sufficiently large so that for a given $\varepsilon > 0$, $\int_a^b |f_{p_m} - f_{p_n}|^2 < \varepsilon$. With $n \to \infty$ this implies $\int_a^b |f_{p_m} - f|^2 \leq \varepsilon$ by Fatou's Lemma (Theorem 2.8.5, page 61). Therefore $f \in L^2([a,b])$. Also

$$\int_a^b |f - f_n|^2 \leq \int_a^b |f - f_{p_n}|2 + \int |f_{p_n} - f_n|^2 < 2\varepsilon$$

for n sufficiently large. Therefore $f_n \to f$ under the L^2 norm and so L^2 is complete. □

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