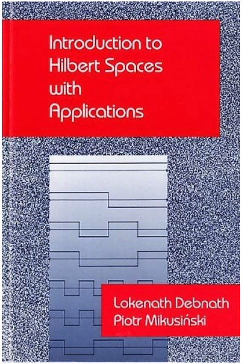
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.6. Strong and Weak Convergence—Proofs of Theorems



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$$|(x_n - x, y)| \leq \|x_n - x\| \, \|y\| \to 0 \text{ as } n \to \infty$$

and so $(x_n - x, y) \to 0$ as $n \to \infty$, for all $y \in E$. Therefore $x_n \xrightarrow{w} x$. □

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Theorem 3.6.2

Theorem 3.6.2. If $x_n \rightarrow_w x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Proof. If $(x_n, y) \to (x, y)$ for all $y \in E$, then $(x_n, x) \to (x, x) = \|x\|^2$. Then

$$\|x_n - x\|^2 = (x_n - x, x_n - x) = (x_n, x_n) = (x_n, x) - (x, x_n) + (x_n, x_n)$$

$$= \|x_n\|^2 - 2\text{Re}(x_n, x) + \|x\|^2 \to \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0$$

as $n \to \infty$. □

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