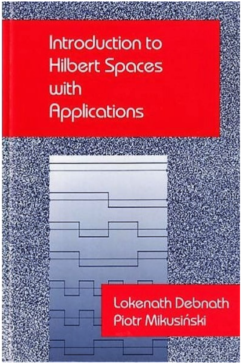
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.7. Orthogonal and Orthonormal Systems—Proofs of Theorems



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Proof. Let S be an orthogonal system. Suppose $\sum_{k=1}^{n} \alpha_k x_k = 0$ for scalars $\alpha_k \in \mathbb{C}$. Then

$$0 = \left(\sum_{k=1}^{n} \alpha_k x_k, \sum_{k=1}^{n} \alpha_k x_k \right) = \sum_{k=1}^{n} |\alpha_k|^2 \|x_k\|^2.$$

Therefore $\alpha_k = 0$ for all $k \in \mathbb{N}$ and so any finite subset of S is linearly independent and so S is linearly independent. □

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Example 3.7.3. Legendre Polynomials

Example 3.7.3. The *Legendre polynomials* defined by $P_0(x) = 1$,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n], \text{ for } n \in \mathbb{N}$$

form an orthogonal system in $L^2([-1, 1])$.

Solution. Denote $p_n(x) = (x^2 - 1)^n$. Then

$$\int_{-1}^{1} P_n(x) x^m \, dx = \frac{1}{2^n n!} \int_{-1}^{1} p_n^{(n)}(x) x^m \, dx.$$

Notice that for $x = \pm 1$ and $k = 0, 1, \ldots, (n-1)$ that $p_n^{(k)}(x) = 0$.

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$$\int_{-1}^{1} p_n^{(n)}(x) x^m \, dx = x^m p_n^{(n-1)}(x)\Big|_{-1}^{1} - \int_{-1}^{1} m x^{m-1} p_n^{(n-1)}(x)\, dx \ldots$$

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Example 3.7.3 (continued 1)

Solution (continued). ...

$$= -m \int_{-1}^{1} p_n^{(n-1)}(x) x^{m-1}\, dx.$$

Repeated Integration by Parts yields

$$\int_{-1}^{1} p_n^{(n)}(x) x^m\, dx = (-1)^m m! \int_{-1}^{1} p_n^{(n-m)}(x)\, dx$$

$$= (-1)^m m! (p_n^{(n-m-1)}(x))|_{-1}^{1} = 0 \ (m < n).$$

Therefore $\int_{-1}^{1} P_n(x) x^m\, dx = 0$ for $m < n$ and $P_n(x)$ is orthogonal to x^m for all $m < n$. Since P_m is a polynomial of degree m,

$$(P_n, P_m) = \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \text{ for } m \neq n.$$

Therefore the Legendre polynomials form an orthogonal system.

Example 3.7.3 (continued 1)

Solution (continued). ...

$$= -m \int_{-1}^{1} p_n^{(n-1)}(x) x^{m-1}\, dx.$$

Repeated Integration by Parts yields

$$\int_{-1}^{1} p_n^{(n)}(x) x^m\, dx = (-1)^m m! \int_{-1}^{1} p_n^{(n-m)}(x)\, dx$$

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Therefore $\int_{-1}^{1} P_n(x) x^m\, dx = 0$ for $m < n$ and $P_n(x)$ is orthogonal to x^m for all $m < n$. Since P_m is a polynomial of degree m,

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Example 3.7.3 (continued 2)

Example 3.7.3. The *Legendre polynomials* defined by $P_0(x) = 1$,

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form an orthogonal system in $L^2([-1, 1])$.

Solution (continued). Notice that $\displaystyle\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{2}{2n+1}$ (see page 102) and so $\sqrt{n + 1/2} P_n(x)$ form an orthonormal system in $L^2([-1, 1])$.