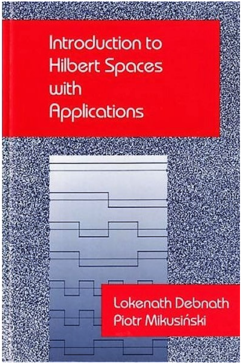
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.8. Properties of Orthonormal Systems—Proofs of Theorems



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$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

and $\sum_{k=1}^{n} |(x, x_k)|^2 \leq \|x\|^2$.

Proof. We have for arbitrary $\alpha_1, \alpha_2, \dots \in \mathbb{C}$,

$$\left\| x - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 = \left(x - \sum_{k=1}^{\infty} \alpha_k x_k, x - \sum_{k=1}^{\infty} \alpha_k x_k \right)$$

$$= \|x\|^2 - \left(x, \sum_{k=1}^{\infty} \alpha_k x_k \right) - \left(\sum_{k=1}^{\infty} \alpha_k x_k, x \right) + \sum_{k=1}^{\infty} |\alpha_k|^2 \dots$$

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Proof (continued). ...

$$= \|x\|^2 - \sum_{k=1}^{\infty} \overline{\alpha}_k (x, x_k) - \sum_{k=1}^{\infty} \alpha_k \overline{(x, x_k)} + \sum_{k=1}^{\infty} \alpha_k \overline{\alpha}_k$$

$$= \|x\|^2 - \sum_{k=1}^{\infty} |(x, x_k)|^2 + \sum_{k=1}^{\infty} |(x, x_k) - \alpha_k|^2.$$

If $\alpha_k = (x, x_k)$ the first claim holds. Since

$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 \geq 0,$$

then the second claim holds. □

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Theorem 3.8.3

Theorem 3.8.3. Let $\{x_n\}$ be an orthonormal sequence is a Hilbert space and let $\{\alpha_n\} \subset \mathbb{C}$ be a sequence. Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\{\alpha_n\} \in \ell^2$. Then

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \sqrt{\sum_{n=1} \infty |\alpha_n|^2}.$$

Proof. For every $m > k > 0$ we have

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \sqrt{\sum_{n=1}^{\infty} |\alpha_n|^2}. \tag{3.8.8}$$

by the Pythagorean Formula (Theorem 3.8.1). If $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ then $s_m = \sum_{n=1}^{m} \alpha_n x_n$ is a Cauchy sequence in the Hilbert space and therefore the series converges.

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Proof. Suppose $\{x_n\}$ is complete in H. Then for all $x \in H$ we have $x = \sum_{n=1}^{\infty}(x, x_n)x_n$. Therefore if $(x, x_n) = 0$ for all $n \in \mathbb{N}$ then $x = 0$. Conversely suppose $(x, x_n) = 0$ for all $n \in \mathbb{N}$ implies $x = 0$. Let $x \in H$.

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$$(x - x, x_n) = (x, x_n) - \left(\sum_{k=1}^{\infty}(x, x_k)x_k, x_n\right)$$

$$= (x, x_n) - \sum_{k=1}^{\infty}(x, x_k)(x_k, x_n) = (x, x_n) - (x, x_n) = 0,$$

and so $x - y = 0$ and hence $x = \sum_{n=1}^{\infty}(x, x_n)x_n$. □

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An orthonormal sequence $\{x_n\}$ in a Hilbert space H is complete if and only if

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$$\|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2 \to 0 \text{ as } n \to \infty.$$

Thus

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