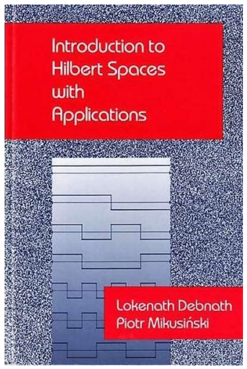
Advanced Differential Equations

Chapter 4. Linear Operators on Hilbert Spaces

Section 4.11. The Fourier Transform—Proofs of Theorems



Theorem 4.11.2

Theorem 4.11.2. The Fourier transform of an integrable function is a continuous function.

Proof. Let $f \in L^1(\mathbb{R})$.For any $k, h \in \mathbb{R}$ we have

$$|\hat{f}(k+h) - \hat{f}(k)| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{-i(k+h)x} f(x) - e^{-ikx} f(x))\, dx \right|$$
$$= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{-ikx}(e^{-ihx} - 1) f(x)\, dx \right|$$
$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx} - 1||f(x)|\, dx \text{ since } |e^{-ikx}| = 1. \quad (4.11.3)$$

Now $|e^{-ihx} - 1||f(x)| \leq 2|f(x)|$ where f is integrable an $\lim_{h\to 0} |e^{-ihx} - 1| = 0$ for all $x \in \mathbb{R}$, so $\lim_{h\to 0} \frac{1}{\sqrt{2\pi}} |e^{-ihx} - 1||f(x)|\, dx$ by the Lebesgue Dominated Convergence Theorem, Theorem 2.8.4. That is, $\lim_{h\to 0} \hat{f}(k+h) = \hat{f}(k)$ and $\mathscr{F}(f) = \hat{f}$ is continuous at k. Since k is an arbitrary real number, then $\mathscr{F}(f) = \hat{f}$ is continuous on $\mathbb{R}$. □

Theorem 4.11.3

Theorem 4.11.3. If $f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = \|f_n - f\|_1 \to 0$ as $n \to \infty$ then the sequence of Fourier transforms $\{\hat{f}_n\}$ converges to $\hat{f}$ uniformly on $\mathbb{R}$.

Proof. First,

$$|\hat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx} f(x)|\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx$$

for all $k \in \mathbb{R}$. So

$$\sup_{k\in\mathbb{R}} |\hat{f}_n(k) - \hat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = \frac{1}{\sqrt{2\pi}} \|f_n - f\|_1.$$

Since $\|f_n - f\|_2 \to 0$ then $\sup_{k\in\mathbb{R}} |\hat{f}_n(k) = \hat{f}(k)| \to 0$ as $n \to \infty$ and hence $\hat{f}_n \to \hat{f}$ uniformly on $\mathbb{R}$. □

Theorem 4.11.4

Theorem 4.11.4. The Riemann-Lebesgue Theorem. If $f \in L^1(\mathbb{R})$ then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Proof. Since $e^{-ikx} = -(-1)e^{-ikx} = -e^{-i\pi}e^{-ikx} = -e^{-ikx-i\pi}$ then

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^{-ikx-i\pi} f(x)\, dx = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x+\pi/k)} f(x)\, dx$$
$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x - \pi/k)\, dx.$$

Hence

$$\hat{f}(k) = \frac{1}{2}(\hat{f}(k) + \hat{f}(k))$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x - \pi/k)\, dx \right)$$
$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}(f(x) - f(x - \pi/k))\, dx \ldots$$

Theorem 4.11.4 (continued)

Theorem 4.11.4. The Riemann-Lebesgue Theorem.
If $f \in L^1(\mathbb{R})$ then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Proof (continued).
...and so $|\hat{f}(k)| \leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x) - f(x - \pi/k)|\, dx$. Theorem 2.4.2 states: "If $f \in L^1(\mathbb{R})$, the $\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|\, dx = 0$." Since $f(x) - f(x - \pi/k) \in L^1(\mathbb{R})$, then by Theorem 2.4.2 $\lim_{k\to\infty} \int_{-\infty}^{\infty} |f(x + \pi/k) - f(x)| = 0$. Therefore, $\lim_{k\to\infty} |\hat{f}(k)| = 0$, as claimed. □

Theorem 4.11.6

Theorem 4.11.6. If f is a continuous piecewise differentiable function, $f, f' \in L^1(\mathbb{R})$, and $\lim_{|x|\to\infty} f(x) = 0$ then $\mathscr{F}\{f'\} = ik\mathscr{F}\{f\}$.

Proof. Integration by parts gives

$$\begin{aligned}
\mathscr{F}\{f'\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} f(x)e^{-ikx}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik)f(x)e^{-ikx}\, dx \\
&= 0 + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-kx} f(x)\, dx \text{ since } \lim_{|x|\to\infty} f(x) = 0 \\
&\quad \text{because } f \in L^1(\mathbb{R}) \text{ and } f \text{ is continuous} \\
&= ik\mathscr{F}\{f\}.
\end{aligned}$$

□

Theorem 4.11.7

Theorem 4.11.7. Convolution Theorem.
Let $f, g \in L^1(\mathbb{R})$. Then $\mathscr{F}\{f * g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$.
Proof. Let $f, g \in L^1(\mathbb{R})$ and $h = f * g$. Then $h \in L^1(\mathbb{R})$ by Theorem 2.15.1 (the proof of which is based on Fubini's Theorem) and so $\mathscr{F}(h)$ is defined. We have

$$\begin{aligned}
\hat{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u)g(u)\, du \right) e^{-ikx}\, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\, du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} e^{-ik(x+u)} f(x)\, dx\, du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iku} g(u)\, du \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \hat{g}(k)\hat{f}(k).
\end{aligned}$$

□

Theorem 4.11.8

Theorem 4.11.8. Let f be a continuous function on $\mathbb{R}$ vanishing outside a bounded interval. Then $\hat{f} = \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \|f\|_2$.

Proof. First, suppose f vanishes outside the interval $[-\pi, \pi]$. The sequence of functions $\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx}$ for $n \in \mathbb{Z}$ is an orthonormal sequence in $L^2([-\pi, \pi])$ (and also in $L^2(\mathbb{R})$), so by Parseval's Formula (Theorem 3.8.5) we get

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-inx} f(x)\, dx \right|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Replacing $f(x)$ with $e^{-\xi x} f(x)$ in the previous equality we get $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n+\xi)|^2$ (since $\|f\|_2^2 = \|e^{-i\xi x} f(x)\|_2^2$).

Theorem 4.11.8 (continued 1)

Proof (continued). Integration of both sides with respect to ξ from 0 to 1 yields

$$\begin{aligned}
\|f\|_2^2 &= = \sum_{n=-\infty}^{\infty} \int_0^1 |\hat{f}(n+\xi)|^2\, d\xi \\
&= \cdots + \int_0^1 |\hat{f}(-1+\xi)|^2\, d\xi + \int_0^1 |\hat{f}(0+\xi)|^2\, d\xi \\
&\quad + \int_0^1 |\hat{f}(1+\xi)|^2\, d\xi \cdots \\
&= \cdots + \int_{-1}^0 |\hat{f}(\xi)|^2\, d\xi + \int_0^1 |\hat{f}(\xi)|^2\, d\xi + \int_1^2 |\hat{f}(\xi)|^2\, d\xi + \cdots \\
&= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \|f\|_2^2,
\end{aligned}$$

as claimed (in the case that f vanishes outside $[-\pi, \pi]$).

Theorem 4.11.8 (continued 2)

Theorem 4.11.8. Let f be a continuous function on $\mathbb{R}$ vanishing outside a bounded interval. Then $\hat{f} = \in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \|f\|_2$.

Proof (continued). If f does not vanish outside $[-\pi, \pi]$, then we take a positive λ for which $g(x) = f(\lambda x)$ vanishes outside $[-\pi, \pi]$. Then $\hat{g}(k) = (1/\lambda)\hat{f}(k/\lambda)$ and, as argued above,

$$\|f\|_2^2 = \lambda\|g\|_2^2 = \lambda\|\hat{g}\|_2^2 = \lambda \int_{-\infty}^{\infty} \left|\frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)\right|^2 d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \|\hat{f}\|_2^2,$$

as claimed in the general case. □

Theorem 4.11.9

Theorem 4.11.9. Parseval's Relation.
If $f \in L^2(\mathbb{R})$ then $\|\hat{f}\|_2 = \|f\|_2$.

Proof. Let $\{\varphi_n\}$ be a sequence of continuous functions with compact support convergent to f in $L^2(\mathbb{R})$. Then by Theorem 4.11.8, $\|\hat{\varphi}_n\|_2 = \|\varphi_n\|_2$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned}
\|\hat{f}\|_2 &= \left\|\lim_{n\to\infty} \hat{\varphi}_n\right\|_2 \\
&= \lim_{n\to\infty} \|\hat{\varphi}_n\|_2 \text{ since } \|\cdot\|_2 \text{ is a continuous mapping into } \mathbb{R} \\
&= \lim_{n\to\infty} \|\varphi_n\|_2 \\
&= \left\|\lim_{n\to\infty} \varphi_n\right\|_2 \text{ since } \|\cdot\|_2 \text{ is a continuous mapping into } \mathbb{R} \\
&= \|f\|_2,
\end{aligned}$$

as claimed. □

Theorem 4.11.10

Theorem 4.11.10. Let $f \in L^2(\mathbb{R})$. Then

$$\hat{f}(k) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Proof. For $n \in \mathbb{N}$ define $f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n. \end{cases}$ Then $|f - f_n\|_2 \to 0$ and so $\|\hat{f} - \hat{f}_n\|_2 \to 0$ as $n \to \infty$. Also,

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f_n(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx$$

since $f_n(x) = 0$ for $|x| \geq n$ and the claim follows. □

Theorem 4.11.11

Theorem 4.11.11. Weak Parseval's Relation.
If $f, g \in L^2(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx.$$

Proof. For $n \in \mathbb{Z}$ define

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \ge n \end{cases} \text{ and } g_n(x) = \begin{cases} g(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \ge n. \end{cases}$$

Now $\hat{f}_m(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ix\xi} f_m(\xi)\,d\xi$, so

$$\int_{-\infty}^{\infty} \hat{f}_m(x)g_n(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_n(x) \int_{-\infty}^{\infty} e^{-in\xi} f_m(\xi)\,d\xi\,dx.$$

Theorem 4.11.11 (continued 1)

Proof (continued). The function of x and ξ, $e^{-inx}g_n(x)f_m(x)$ is integrable over $\mathbb{R}^2$. So Fubini's Theorem (Theorem 2.14.1, which allows us to change the order of integration in a double integral, with $f(x,y) = e^{-in\xi}g_n(x)f_m(\xi)$ and $F(x) = \int_{-\infty}^{\infty} e^{-in\xi}g_x(x)f_m(\xi)\,d\xi$) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = \int_{-\infty}^{\infty} F(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-in\xi}g_n(x)f_m(\xi)\,d\xi\,dx < \infty$$

and so

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}_m g_n(x)\,dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_n(x)\int_{-\infty}^{\infty} e^{-ix\xi}f_m(\xi)\,d\xi\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix\xi}g_n(x)f_m(\xi)\,d\xi\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix\xi}g_n(x)f_m(\xi)\,dxd\xi\end{aligned}$$

Theorem 4.11.11 (continued 2)

Proof (continued). ...

$$\int_{-\infty}^{\infty} \hat{f}_m g_n(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_m(\xi)\int_{-\infty}^{\infty} e^{-ix\xi}g_n(x)\,dx\,d\xi = \int_{-\infty}^{\infty} f_m(\xi)\hat{g}(\xi)\,d\xi.$$

Since $\|g - g_n\|_2 \to 0$ and $\|\hat{g} - \hat{g}_n\| \to 0$, then be letting $n \to \infty$ we have

$$\int_{-\infty}^{\infty} \hat{f}_m(x)g(x)\,dx = \int_{-\infty}^{\infty} f_m(x)\hat{g}(x)\,dx$$

by the continuity of the inner product (or continuity of the norm $\|\cdot\|_2$). Similarly, letting $m \to \infty$ we have

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx,$$

as claimed. □

Lemma 4.11.1

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Proof. Using the inner product on $L^2(\mathbb{R})$ we have

$$\begin{aligned}(f, \overline{\hat{g}}) &= \int_{-\infty}^{\infty} f(x)\overline{\hat{g}}(x)\,dx \\ &= \int_{-\infty}^{\infty} \hat{f}(x)\overline{g}(x)\,dx \text{ by Theorem 4.11.11, Weak Parseval's Relation} \\ &= \int_{-\infty}^{\infty} \hat{f}(x)\hat{f}(x)\,dx \text{ since } g = \overline{\hat{f}} \text{ or } \overline{g} = \hat{f} \text{ by hypothesis} \\ &= (\hat{f}, \hat{f}) = \|\hat{f}\|_2^2 \\ &= \|f\|_2^2 \text{ by Theorem 4.11.9, Parseval's Relation.}\end{aligned}$$

Since $\|f\|_2^2$ is real then $\overline{(f, \overline{\hat{g}})} = \|f\|_2^2$. By Theorem 4.11.9 (Parseval's Relation), $\|\hat{g}\|_2^2 = \|g\|_2^2$ and $\|\hat{f}\|_2^2 = \|f\|_2^2$, and since $g = \overline{\hat{f}}$ by hypothesis then $\|g\|_2 = \|\overline{\hat{f}}\|_2 = \|\hat{f}\|_2$ so that $\|\hat{g}\|_2^2 = \|g\|_2^2 = \|\hat{f}\|_2^2 = \|f\|_2^2$.

Lemma 4.11.1 (continued)

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Proof (continued). Finally,

$$\|f - \overline{\hat{g}}\|_2^2 = (f - \overline{\hat{g}}, f - \overline{\hat{g}}) = (f, f) - (\overline{\hat{g}}, f) - (f, \overline{\hat{g}}) + (\overline{\hat{g}}, \overline{\hat{g}})$$

$$= \|f\|_2^2 - \overline{(f, \overline{\hat{g}})} - (f, \overline{\hat{g}}) + \|\overline{\hat{g}}\|_2^2 = \|f\|_2^2 - \|f\|_2^2 - \|f\|_2^2 + \|f\|_2^2 = 0$$

since $\overline{(f, \overline{\hat{g}})} = \|f\|_2^2$, $(f, \overline{\hat{g}}) = \|f\|_2^2$, and $\|\overline{\hat{g}}\|_2 = \|\hat{g}\|_2 = \|f\|_2$. Therefore, $f = \overline{\hat{g}}$, as claimed. □

Theorem 4.11.12

Theorem 4.11.12. Inversion of Fourier Transform on $L^2(\mathbb{R})$.
Let $f \in L^2(\mathbb{R})$. Then

$$f(x) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(k)\, dk$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$. Define $g = \overline{\hat{f}}$. Then by Lemma 4.11.1,

$$f(x) = \overline{\hat{g}}(x) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(k)\, dk} = \overline{\frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-ikx} g(k)\, dk}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{ikx} \overline{g}(k)\, dk = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-kx} \hat{f}(k)\, dk,$$

as claimed. □

Theorem 4.11.13

Theorem 4.11.13. General Persaval's Relation.
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\, dk.$$

Proof. By the polarization identity states (see Exercise 3.13.9)

$$(f, g) = \frac{1}{4}(\|f + g\|_2^2 - \|f - g\|_2^2 + i\|f_i g\|_2^2 - i\|f - ig\|_2^2).$$

So

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx &= (f, g) \text{ by definition of inner product on } L^2(\mathbb{R}) \\ &= \frac{1}{4}(\|f + g\|_2^2 - \|f - g\|_2^2 + i\|f - ig\|_2^2 - i\|f - ig\|_2^2) \\ &\quad \text{by the polarization identity}\end{aligned}$$

Theorem 4.11.13 (continued)

Theorem 4.11.13. General Persaval's Relation.
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\, dk.$$

Proof (continued). ...

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx &= \frac{1}{4}(\|\hat{f} + \hat{g}\|_2^2 - \|\hat{f} - \hat{g}\|_2^2 + i\|\hat{f} - i\hat{g}\|_2^2 - i\|\hat{f} - i\hat{g}\|_2^2) \\ &\quad \text{since } \mathscr{F} \text{ is linear (by Theorem 4.11.1)} \\ &\quad \text{and by Theorem 4.11.9, Parseval's Relation} \\ &= (\hat{f}, \hat{g}) \text{ by the polarization identity} \\ &= \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}}(x)\, dx \text{ by the definition of inner product,}\end{aligned}$$

as claimed. □

Theorem 4.11.14

Theorem 4.11.14. Plancherel's Theorem.
For every $f \in L^2(\mathbb{R})$ there exists $\hat{f} \in L^2(\mathbb{R})$ such that:

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$.

(b) $\left\| \hat{f}(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\,dx \right\|_2 \to 0$ and
$\left\| f(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(x)\,dx \right\|_2 \to 0$ as $n \to \infty$.

(c) $\|f\|_2 = \|\hat{f}\|_2$.

(d) The mapping $f \mapsto \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Proof. We still need to prove part (d).
We know that $\mathscr{F}$ is linear by Theorem 4.11.1. We know that the mapping preserves inner products by the General Parseval's Relation, Theorem 4.11.13.

Theorem 4.11.14. Plancherel's Theorem

Theorem 4.11.14 (continued)

Proof (continued). If $\hat{f} = \hat{g}$ then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\,dx$$

or $\int_{-\infty}^{\infty} e^{-ikx}(f(x) - g(x))\,dx = 0$ for all $k \in \mathbb{R}$, and hence $f(x) = g(x)$ almost everywhere (i.e., $f = g$ in $L^2(\mathbb{R})$). For onto, let $f \in L^2(\mathbb{R})$ and define $h = \overline{f}$ and $g = \overline{\hat{h}}$. By Lemma 4.11.1, $\overline{f} = h = \overline{\hat{g}}$, so that $f = \hat{g}$. So f is the image of g under the mapping and the mapping is onto. That is, $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert space isomorphism, as claimed. □

Theorem 4.11.15

Theorem 4.11.15. The Fourier transform is an unitary operator on $L^2(\mathbb{R})$.

Proof. First, note that $\mathscr{F}\{\overline{g}\}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \overline{g}(x)\,dx$

$$= \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} g(x)\,dx} = \overline{\mathscr{F}^{-1}\{g\}(k)}. \quad (*)$$

Then

$$\begin{aligned}
(\mathscr{F}\{f\}, g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}\{f\}\overline{g}(x)\,dx \text{ by the definition of inner product} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathscr{F}\{\overline{g}\}(x)\,dx \text{ by Theorem 4.11.11} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{\mathscr{F}^{-1}\{g\}(k)}\,dx \text{ by } (*) \\
&= (f, \mathscr{F}^{-1}\{g\}) \text{ by the definition of inner product.}
\end{aligned}$$

So $\mathscr{F}^*\mathscr{F}^{-1}$ and $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathcal{I}$ so that $\mathscr{F}$ is unitary, as claimed. □