Advanced Differential Equations

Chapter 4. Linear Operators on Hilbert Spaces

Section 4.11. The Fourier Transform—Proofs of Theorems

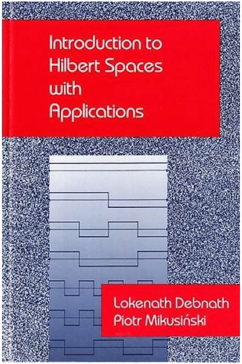

Table of contents

Theorem 4.11.2

Theorem 4.11.2. The Fourier transform of an integrable function is a continuous function.

Proof. Let $f \in L^1(\mathbb{R})$.For any $k, h \in \mathbb{R}$ we have

$$\begin{aligned} |\hat{f}(k+h) - \hat{f}(k)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{-i(k+h)x} f(x) - e^{-ikx} f(x))\, dx \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{-ikx} (e^{-ihx} - 1) f(x)\, dx \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx} - 1||f(x)|\, dx \text{ since } |e^{-ikx}| = 1. \end{aligned} \tag{4.11.3}$$

Theorem 4.11.2

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Now $|e^{-ihx} - 1||f(x)| \leq 2|f(x)|$ where f is integrable an $\lim_{h\to 0} |e^{-ihx} - 1| = 0$ for all $x \in \mathbb{R}$, so $\lim_{h\to 0} \frac{1}{\sqrt{2\pi}} |e^{-ihx} - 1||f(x)|\, dx$ by the Lebesgue Dominated Convergence Theorem, Theorem 2.8.4. That is, $\lim_{h\to 0} \hat{f}(k+h) = \hat{f}(k)$ and $\mathscr{F}(f) = \hat{f}$ is continuous at k. Since k is an arbitrary real number, then $\mathscr{F}(f) = \hat{f}$ is continuous on $\mathbb{R}$. □

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Theorem 4.11.3

Theorem 4.11.3. If $f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = \|f_n - f\|_1 \to 0$ as $n \to \infty$ then the sequence of Fourier transforms $\{\hat{f}_n\}$ converges to $\hat{f}$ uniformly on $\mathbb{R}$.

Proof. First,

$$|\hat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx} f(x)|\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx$$

for all $k \in \mathbb{R}$. So

$$\sup_{k\in\mathbb{R}} |\hat{f}_n(k) - \hat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = \frac{1}{\sqrt{2\pi}} \|f_n - f\|_1.$$

Since $\|f_n - f\|_2 \to 0$ then $\sup_{k\in\mathbb{R}} |\hat{f}_n(k) = \hat{f}(k)| \to 0$ as $n \to \infty$ and hence $\hat{f}_n \to \hat{f}$ uniformly on $\mathbb{R}$. □

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Theorem 4.11.4

Theorem 4.11.4. The Riemann-Lebesgue Theorem.
If $f \in L^1(\mathbb{R})$ then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Proof. Since $e^{-ikx} = -(-1)e^{-ikx} = -e^{-i\pi}e^{-ikx} = -e^{-ikx-i\pi}$ then

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^{-ikx-i\pi} f(x)\,dx = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x+\pi/k)} f(x)\,dx$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x - \pi/k)\,dx.$$

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Hence

$$\hat{f}(k) = \frac{1}{2}(\hat{f}(k) + \hat{f}(k))$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x-\pi/k)\,dx\right)$$

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Theorem 4.11.4 (continued)

Theorem 4.11.4. The Riemann-Lebesgue Theorem.
If $f \in L^1(\mathbb{R})$ then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Proof (continued).
...and so $|\hat{f}(k)| \leq \dfrac{1}{2\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} |f(x) - f(x - \pi/k)|\,dx$. Theorem 2.4.2 states: "If $f \in L^1(\mathbb{R})$, the $\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|\,dx = 0$." Since $f(x) - f(x - \pi/k) \in L^1(\mathbb{R})$, then by Theorem 2.4.2 $\displaystyle\lim_{k\to\infty} \int_{-\infty}^{\infty} |f(x + \pi/k) - f(x)| = 0$. Therefore, $\lim_{k\to\infty} |\hat{f}(k)| = 0$, as claimed. $\square$

Theorem 4.11.6

Theorem 4.11.6. If f is a continuous piecewise differentiable function, $f, f' \in L^1(\mathbb{R})$, and $\lim_{|x|\to\infty} f(x) = 0$ then $\mathscr{F}\{f'\} = ik\mathscr{F}\{f\}$.

Proof. Integration by parts gives

$$\begin{aligned}
\mathscr{F}\{f'\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} f(x)e^{-ikx}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-ik)f(x)e^{-ikx}\,dx \\
&= 0 + \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx} f(x)\,dx \text{ since } \lim_{|x|\to\infty} f(x) = 0 \\
&\quad \text{because } f \in L^1(\mathbb{R}) \text{ and } f \text{ is continuous} \\
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Theorem 4.11.7

Theorem 4.11.7. Convolution Theorem.
Let $f, g \in L^1(\mathbb{R})$. Then $\mathscr{F}\{f * g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$.

Proof. Let $f, g \in L^1(\mathbb{R})$ and $h = f * g$. Then $h \in L^1(\mathbb{R})$ by Theorem 2.15.1 (the proof of which is based on Fubini's Theorem) and so $\mathscr{F}(h)$ is defined. We have

$$\begin{aligned}
\hat{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u) g(u)\, du \right) e^{-ikx}\, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\, du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} e^{-ik(x+u)} f(x)\, dx\, du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iku} g(u)\, du \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \hat{g}(k)\hat{f}(k). \qquad \square
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Theorem 4.11.8

Theorem 4.11.8. Let f be a continuous function on $\mathbb{R}$ vanishing outside a bounded interval. Then $\hat{f} =\in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \|f\|_2$.

Proof. First, suppose f vanishes outside the interval $[-\pi, \pi]$. The sequence of functions $\varphi_n(x) = \dfrac{1}{\sqrt{2\pi}} e^{-inx}$ for $n \in \mathbb{Z}$ is an orthonormal sequence in $L^2([-\pi, \pi])$ (and also in $L^2(\mathbb{R})$), so by Parseval's Formula (Theorem 3.8.5) we get

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-inx} f(x)\, dx \right|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Replacing $f(x)$ with $e^{-\xi x} f(x)$ in the previous equality we get $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n+\xi)|^2$ (since $\|f\|_2^2 = \|e^{-i\xi x} f(x)\|_2^2$).

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Theorem 4.11.8 (continued 1)

Proof (continued). Integration of both sides with respect to ξ from 0 to 1 yields

$$\begin{aligned}
\|f\|_2^2 &= \; = \sum_{n=-\infty}^{\infty} \int_0^1 |\hat{f}(n+\xi)|^2 \, d\xi \\
&= \cdots + \int_0^1 |\hat{f}(-1+\xi)|^2 \, d\xi + \int_0^1 |\hat{f}(0+\xi)|^2 \, d\xi \\
&\quad + \int_0^1 |\hat{f}(1+\xi)|^2 \, d\xi \cdots \\
&= \cdots + \int_{-1}^0 |\hat{f}(\xi)|^2 \, d\xi + \int_0^1 |\hat{f}(\xi)|^2 \, d\xi + \int_1^2 |\hat{f}(\xi)|^2 \, d\xi + \cdots \\
&= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \|f\|_2^2,
\end{aligned}$$

as claimed (in the case that f vanishes outside $[-\pi, \pi]$).

Theorem 4.11.8 (continued 2)

Theorem 4.11.8. Let f be a continuous function on $\mathbb{R}$ vanishing outside a bounded interval. Then $\hat{f} =\in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \|f\|_2$.

Proof (continued). If f does not vanish outside $[-\pi, \pi]$, then we take a positive λ for which $g(x) = f(\lambda x)$ vanishes outside $[-\pi, \pi]$. Then $\hat{g}(k) = (1/\lambda)\hat{f}(k/\lambda)$ and, as argued above,

$$\|f\|_2^2 = \lambda\|g\|_2^2 = \lambda\|\hat{g}\|_2^2 = \lambda \int_{-\infty}^{\infty} \left| \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)\right|^2 d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \|\hat{f}\|_2^2,$$

as claimed in the general case. □

Theorem 4.11.9

Theorem 4.11.9. Parseval's Relation.
If $f \in L^2(\mathbb{R})$ then $\|\hat{f}\|_2 = \|f\|_2$.

Proof. Let $\{\varphi_n\}$ be a sequence of continuous functions with compact support convergent to f in $L^2(\mathbb{R})$. Then by Theorem 4.11.8, $\|\hat{\varphi}_n\|_2 = \|\varphi_n\|_2$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned}
\|\hat{f}\|_2 &= \left\| \lim_{n\to\infty} \hat{\varphi}_n \right\|_2 \\
&= \lim_{n\to\infty} \|\hat{\varphi}_n\|_2 \text{ since } \|\cdot\|_2 \text{ is a continuous mapping into } \mathbb{R} \\
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as claimed. □

Theorem 4.11.10

Theorem 4.11.10. Let $f \in L^2(\mathbb{R})$. Then

$$\hat{f}(k) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Proof. For $n \in \mathbb{N}$ define $f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n. \end{cases}$ Then $\|f - f_n\|_2 \to 0$ and so $\|\hat{f} - \hat{f}_n\|_2 \to 0$ as $n \to \infty$. Also,

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f_n(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx$$

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Theorem 4.11.11

Theorem 4.11.11. Weak Parseval's Relation.
If $f, g \in L^2(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx.$$

Proof. For $n \in \mathbb{Z}$ define

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n \end{cases} \text{ and } g_n(x) = \begin{cases} g(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n. \end{cases}$$

Now $\hat{f}_m(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ix\xi} f_m(\xi)\,d\xi$, so

$$\int_{-\infty}^{\infty} \hat{f}_m(x)g_n(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_n(x) \int_{-\infty}^{\infty} e^{-in\xi} f_m(\xi)\,d\xi\,dx.$$

Theorem 4.11.11

Theorem 4.11.11. Weak Parseval's Relation.
If $f, g \in L^2(\mathbb{R})$ then

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Proof. For $n \in \mathbb{Z}$ define

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Theorem 4.11.11 (continued 1)

Proof (continued). The function of x and ξ, $e^{-inx}g_n(x)f_m(x)$ is integrable over $\mathbb{R}^2$. So Fubini's Theorem (Theorem 2.14.1, which allows us to change the order of integration in a double integral, with $f(x,y) = e^{-in\xi}g_n(x)f_m(\xi)$ and $F(x) = \int_{-\infty}^{\infty} e^{-in\xi}g_x(x)f_m(\xi)\,d\xi$) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = \int_{-\infty}^{\infty} F(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-in\xi}g_n(x)f_m(\xi)\,d\xi\,dx < \infty$$

and so

$$\int_{-\infty}^{\infty} \hat{f}_m g_n(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_n(x)\int_{-\infty}^{\infty} e^{-ix\xi}f_m(\xi)\,d\xi\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix\xi}g_n(x)f_m(\xi)\,d\xi\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix\xi}g_n(x)f_m(\xi)\,dxd\xi$$

Theorem 4.11.11 (continued 2)

Proof (continued). ...

$$\int_{-\infty}^{\infty} \hat{f}_m g_n(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_m(\xi) \int_{-\infty}^{\infty} e^{-ix\xi} g_n(x)\,dx\,d\xi = \int_{-\infty}^{\infty} f_m(\xi)\hat{g}(\xi)\,d\xi.$$

Since $\|g - g_n\|_2 \to 0$ and $\|\hat{g} - \hat{g}_n\| \to 0$, then be letting $n \to \infty$ we have

$$\int_{-\infty}^{\infty} \hat{f}_m(x)g(x)\,dx = \int_{-\infty}^{\infty} f_m(x)\hat{g}(x)\,dx$$

by the continuity of the inner product (or continuity of the norm $\|\cdot\|_2$). Similarly, letting $m \to \infty$ we have

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx,$$

as claimed. □

Lemma 4.11.1

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Proof. Using the inner product on $L^2(\mathbb{R})$ we have

$$\begin{aligned}
(f, \overline{\hat{g}}) &= \int_{-\infty}^{\infty} f(x)\overline{\hat{g}}(x)\,dx \\
&= \int_{-\infty}^{\infty} \hat{f}(x)\overline{g}(x)\,dx \text{ by Theorem 4.11.11, Weak Parseval's Relation} \\
&= \int_{-\infty}^{\infty} \hat{f}(x)\hat{f}(x)\,dx \text{ since } g = \overline{\hat{f}} \text{ or } \overline{g} = \hat{f} \text{ by hypothesis} \\
&= (\hat{f}, \hat{f}) = \|\hat{f}\|_2^2 \\
&= \|f\|_2^2 \text{ by Theorem 4.11.9, Parseval's Relation.}
\end{aligned}$$

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&= (\hat{f}, \hat{f}) = \|\hat{f}\|_2^2 \\
&= \|f\|_2^2 \text{ by Theorem 4.11.9, Parseval's Relation.}
\end{aligned}$$

Since $\|f\|_2^2$ is real then $\overline{(f, \overline{\hat{g}})} = \|f\|_2^2$. By Theorem 4.11.9 (Parseval's Relation), $\|\hat{g}\|_2^2 = \|g\|_2^2$ and $\|\hat{f}\|_2^2 = \|f\|_2^2$, and since $g = \overline{\hat{f}}$ by hypothesis then $\|g\|_2 = \|\overline{\hat{f}}\|_2 = \|\hat{f}\|_2$ so that $\|\hat{g}\|_2^2 = \|g\|_2^2 = \|\hat{f}\|_2^2 = \|f\|_2^2$.

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Lemma 4.11.1 (continued)

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Proof (continued). Finally,

$$\|f - \overline{\hat{g}}\|_2^2 = (f - \overline{\hat{g}}, f - \overline{\hat{g}}) = (f, f) - (\overline{\hat{g}}, f) - (f, \overline{\hat{g}}) + (\overline{\hat{g}}, \overline{\hat{g}})$$

$$= \|f\|_2^2 - \overline{(f, \overline{\hat{g}})} - (f, \overline{\hat{g}}) + \|\overline{\hat{g}}\|_2^2 = \|f\|_2^2 - \|f\|_2^2 - \|f\|_2^2 + \|f\|_2^2 = 0$$

since $\overline{(f, \overline{\hat{g}})} = \|f\|_2^2$, $(f, \overline{\hat{g}}) = \|f\|_2^2$, and $\|\overline{\hat{g}}\|_2 = \|\hat{g}\|_2 = \|f\|_2$. Therefore, $f = \overline{\hat{g}}$, as claimed. □

Lemma 4.11.1 (continued)

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Proof (continued). Finally,

$$\|f - \overline{\hat{g}}\|_2^2 = (f - \overline{\hat{g}}, f - \overline{\hat{g}}) = (f, f) - (\overline{\hat{g}}, f) - (f, \overline{\hat{g}}) + (\overline{\hat{g}}, \overline{\hat{g}})$$

$$= \|f\|_2^2 - \overline{(f, \overline{\hat{g}})} - (f, \overline{\hat{g}}) + \|\overline{\hat{g}}\|_2^2 = \|f\|_2^2 - \|f\|_2^2 - \|f\|_2^2 + \|f\|_2^2 = 0$$

since $\overline{(f, \overline{\hat{g}})} = \|f\|_2^2$, $(f, \overline{\hat{g}}) = \|f\|_2^2$, and $\|\overline{\hat{g}}\|_2 = \|\hat{g}\|_2 = \|f\|_2$. Therefore, $f = \overline{\hat{g}}$, as claimed. $\square$

Theorem 4.11.12

Theorem 4.11.12. Inversion of Fourier Transform on $L^2(\mathbb{R})$.
Let $f \in L^2(\mathbb{R})$. Then

$$f(x) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(k)\, dk$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$. Define $g = \overline{\hat{f}}$. Then by Lemma 4.11.1,

$$f(x) = \hat{\overline{g}}(x) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(k)\, dk} = \overline{\frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-ikx} g(k)\, dk}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{ikx} \overline{g}(k)\, dk = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-kx} \hat{f}(k)\, dk,$$

as claimed. $\square$

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Proof. Let $f \in L^2(\mathbb{R})$. Define $g = \overline{\hat{f}}$. Then by Lemma 4.11.1,

$$f(x) = \overline{\hat{g}}(x) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(k)\, dk} = \overline{\frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-ikx} g(k)\, dk}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{ikx} \overline{g}(k)\, dk = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} e^{-kx} \hat{f}(k)\, dk,$$

as claimed. □

Theorem 4.11.13

Theorem 4.11.13. General Persaval's Relation.
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\,dk.$$

Proof. By the polarization identity states (see Exercise 3.13.9)

$$(f, g) = \frac{1}{4}(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f_i g\|_2^2 - i\|f-ig\|_2^2).$$

So

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx &= (f, g) \text{ by definition of inner product on } L^2(\mathbb{R}) \\
&= \frac{1}{4}(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f-ig\|_2^2 - i\|f-ig\|_2^2) \\
&\quad \text{by the polarization identity}
\end{aligned}$$

Theorem 4.11.13

Theorem 4.11.13. General Persaval's Relation.
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\,dk.$$

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So

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx &= (f, g) \text{ by definition of inner product on } L^2(\mathbb{R}) \\ &= \frac{1}{4}(\|f + g\|_2^2 - \|f - g\|_2^2 + i\|f - ig\|_2^2 - i\|f - ig\|_2^2) \\ &\quad \text{by the polarization identity}\end{aligned}$$

Theorem 4.11.13 (continued)

Theorem 4.11.13. General Persaval's Relation.
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\,dk.$$

Proof (continued). ...

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx &= \frac{1}{4}(\|\hat{f}+\hat{g}\|_2^2 - \|\hat{f}-\hat{g}\|_2^2 + i\|\hat{f}-i\hat{g}\|_2^2 - i\|\hat{f}-i\hat{g}\|_2^2) \\
&\quad \text{since } \mathscr{F} \text{ is linear (by Theorem 4.11.1)} \\
&\quad \text{and by Theorem 4.11.9, Parseval's Relation} \\
&= (\hat{f}, \hat{g}) \text{ by the polarization identity} \\
&= \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}}(x)\,dx \text{ by the definition of inner product,}
\end{aligned}$$

as claimed. □

Theorem 4.11.14

Theorem 4.11.14. Plancherel's Theorem.
For every $f \in L^2(\mathbb{R})$ there exists $\hat{f} \in L^2(\mathbb{R})$ such that:

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{f}(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$.

(b) $\left\| \hat{f}(k) - \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-n}^{n} e^{-ikx} f(x)\, dx \right\|_2 \to 0$ and
$\left\| f(k) - \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-n}^{n} e^{ikx} \hat{f}(x)\, dx \right\|_2 \to 0$ as $n \to \infty$.

(c) $\|f\|_2 = \|\hat{f}\|_2$.

(d) The mapping $f \mapsto \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Proof. We still need to prove part (d).
We know that $\mathscr{F}$ is linear by Theorem 4.11.1. We know that the mapping preserves inner products by the General Parseval's Relation, Theorem 4.11.13.

Theorem 4.11.14

Theorem 4.11.14. Plancherel's Theorem.
For every $f \in L^2(\mathbb{R})$ there exists $\hat{f} \in L^2(\mathbb{R})$ such that:

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{f}(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$.

(b) $$\left\| \hat{f}(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx \right\|_2 \to 0 \text{ and}$$
$$\left\| f(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(x)\, dx \right\|_2 \to 0 \text{ as } n \to \infty.$$

(c) $\|f\|_2 = \|\hat{f}\|_2$.

(d) The mapping $f \mapsto \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Proof. We still need to prove part (d).
We know that $\mathscr{F}$ is linear by Theorem 4.11.1. We know that the mapping preserves inner products by the General Parseval's Relation, Theorem 4.11.13.

Theorem 4.11.14 (continued)

Proof (continued). If $\hat{f} = \hat{g}$ then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\,dx$$

or $\int_{-\infty}^{\infty} e^{-ikx}(f(x) - g(x))\,dx = 0$ for all $k \in \mathbb{R}$, and hence $f(x) = g(x)$ almost everywhere (i.e., $f = g$ in $L^2(\mathbb{R})$). For onto, let $f \in L^2(\mathbb{R})$ and define $h = \overline{f}$ and $g = \overline{\hat{f}}$. By Lemma 4.11.1, $\overline{f} = h = \overline{\hat{g}}$, so that $f = \hat{g}$. So f is the image of g under the mapping and the mapping is onto. That is, $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert space isomorphism, as claimed. □

Theorem 4.11.14 (continued)

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Theorem 4.11.15

Theorem 4.11.15. The Fourier transform is an unitary operator on $L^2(\mathbb{R})$.

Proof. First, note that $\mathscr{F}\{\overline{g}\}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\overline{g}(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}} \overline{\int_{-\infty}^{\infty} e^{ikx} g(x)\,dx} = \overline{\mathscr{F}^{-1}\{g\}(k)}. \quad (*)$$

Then

$$\begin{aligned}
(\mathscr{F}\{f\}, g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}\{f\}\overline{g}(x)\,dx \text{ by the definition of inner product} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathscr{F}\{\overline{g}\}(x)\,dx \text{ by Theorem 4.11.11} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{\mathscr{F}^{-1}\{g\}(k)}\,dx \text{ by } (*) \\
&= (f, \mathscr{F}^{-1}\{g\}) \text{ by the definition of inner product.}
\end{aligned}$$

So $\mathscr{F}^*\mathscr{F}^{-1}$ and $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathcal{I}$ so that $\mathscr{F}$ is unitary, as claimed. □

Theorem 4.11.15

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Then

$$\begin{aligned}
(\mathscr{F}\{f\}, g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}\{f\}\overline{g}(x)\, dx \text{ by the definition of inner product} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathscr{F}\{\overline{g}\}(x)\, dx \text{ by Theorem 4.11.11} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{\mathscr{F}^{-1}\{g\}(k)}\, dx \text{ by } (*) \\
&= (f, \mathscr{F}^{-1}\{g\}) \text{ by the definition of inner product.}
\end{aligned}$$

So $\mathscr{F}^*\mathscr{F}^{-1}$ and $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = \mathcal{I}$ so that $\mathscr{F}$ is unitary, as claimed. $\square$