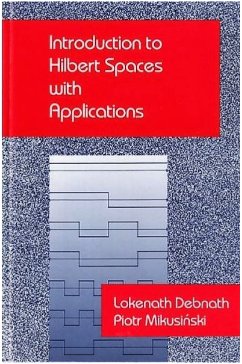
Advanced Differential Equations

Chapter 4. Linear Operators on Hilbert Spaces

Section 4.4. Adjoint and Self Adjoint Operators—Proofs of Theorems



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Theorem 4.4.2. Let A be a bounded linear operator on a Hilbert space. Then operators $T_1 = A^*A$ and $T_2 = A + A^*$ are self adjoint.

Proof. We have

$$(T_1 x, y) = (A^*Ax, y) = Ax, Ay) = (x, A^*Ay),$$

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$$= (x, A * y) + (x, Ay) = (x, A * y + Ay) = (x, (A^* + A)y) = (x, (A + A^*)y),$$

so $T_2 = A + A^*$ is self adjoint. $\square$

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Theorem 4.4.3

Theorem 4.4.3. The product of two self adjoint operators is self adjoint if and only if the operators commute.

Proof. Let A and B be self adjoint. Then

$$(ABx, y) = (Bx, A^*y) = (x, B^*A^*y) = (x, BAy)$$

since $A = A^*$ and $B = B^*$. So if $AB = BA$ then AB is self adjoint.

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Theorem 4.4.4

Theorem 4.4.4. Every bounded linear operator T on a Hilbert space has a representation $T = A + iB$ where A and B are self adjoint. Also, $T^* = A - iB$.

Proof. Let T be a bounded linear operator. Define $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$. By Theorem 4.4.2, A and B are self adjoint. Also

$$(Tx, y) = ((A + iB)x, y) = (Ax, y) + i(Bx, y) = (x, A^*y) + i(x, B^*y)$$

$$= (x, (A^* - iB^*)y) = (x, (A - iB)y).$$

So $T^* = A - iB$. $\square$

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