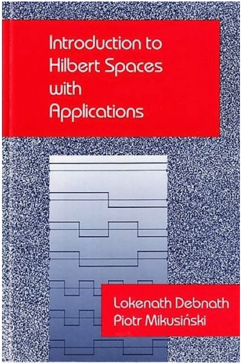
Advanced Differential Equations

Chapter 4. Linear Operators on Hilbert Spaces

Section 4.5. Invertible, Normal, Isometric, and Unitary Operators—Proofs of Theorems



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Theorem 4.5.1

Theorem 4.5.1. The inverse of a linear operator is linear.

Proof. For $x, y \in \mathcal{R}(A)$ and $\alpha, \beta \in \mathbb{C}$ we have

$$A^{-1}(\alpha x + \beta y) = A^{-1}(\alpha A A^{-1} x + \beta A A^{-1} y)$$

$$= A^{-1} A(\alpha A^{-1} x + \beta A^{-1} y) = \alpha A^{-1} x + \beta A^{-1} y.$$

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Theorem 4.5.2

Theorem 4.5.2. Linear operator A is invertible if and only if $Ax = 0$ implies $x = 0$.

Proof. First if A is invertible and $Ax = 0$ then $x = A^{-1}Ax = A^{-1}0 = 0$ (since A^{-1} is linear). Conversely assume $Ax = 0$ implies $x = 0$. If $Ax_1 = Ax_2$ then $A(x_1 - x_2) = 0$ and so $x_1 - x_2 = 0$ and $x_1 = x_2$. Therefore A is one to one and so invertible. $\square$

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Theorem 4.5.9. A bounded linear operator T on a Hilbert space H is isometric if and only if $T^*T = \mathcal{I}$ on H.

Proof. If T is isometric then for all $x \in H$, $\|Tx\|^2 = \|x\|^2$ and so

$$(T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 = \|x\|^2 = (x, x).$$

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