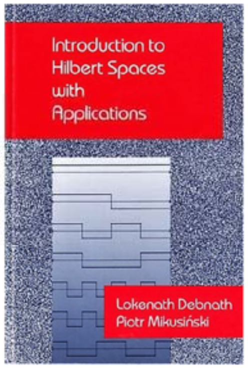
Advanced Differential Equations

Chapter 4. Linear Operators on Hilbert Spaces
Section 4.9. Eigenvalues and Eigenvectors—Proofs of Theorems



Theorem 4.9.2

Theorem 4.9.2. Let T be an invertible linear operator on E and let A be a linear operator on E. Then A and TAT^{-1} have the same eigenvalues.

Proof. Let λ be an eigenvalue of A. Then for some $u \neq 0$ we have $Au = \lambda u$. Since T^{-1} exists, $Tu \neq 0$ and

$$TAT^{-1}\underbrace{(Tu)}_{v} = TAu = T\lambda u = \lambda \underbrace{Tu}_{v}.$$

So λ is an eigenvalue of TAT^{-1}.

Now suppose λ is an eigenvalue of TAT^{-1} Then for some $u \neq 0$ we have $TAT^{-1}u = \lambda u$ wher $u = Tv \in E$. Then

$$Av = T^{-1}(TAT^{-1})(Tv = T^{-1}(TAT^{-1})u = T^{-1}\lambda u = \lambda T^{-1}Tv = \lambda v.$$

So λ is an eigenvalue of A. □

Theorem 4.9.3

Theorem 4.9.3. All eigenvalues of a self adjoint operator on a Hilbert space are real.

Proof. Let λ be an eigenvalue and u an eigenvector of A. Then

$$\lambda(u,u) = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \overline{\lambda}(u,u)$$

implies $\lambda = \overline{\lambda}$ and so λ is real. □

Theorem 4.9.4

Theorem 4.9.4. All eigenvalues of a positive operator are non-negative. All eigenvalues of a strictly positive operator are positive.

Proof. Let A be positive with $Ax = \lambda x$. Then (since A is self adjoint) we have

$$0 \leq (Ax, x) = (\lambda x, x) = \lambda(x,x) = \lambda\|x\|^2.$$

Since $x \neq 0$ then $\lambda \geq 0$. If A is strictly positive, then we can replace "$\leq$" with "$<$." □

Theorem 4.9.5

Theorem 4.9.5. All eigenvalues of a unitary operator on a Hilbert space are complex numbers of modulus 1.

Proof. Let λ be an eigenvalue of unitary operator A with $Au = \lambda u$. Then

$$(Au, Au) = (\lambda u, \lambda u) = |\lambda|^2 \|u\|^2.$$

Also

$$(Au, Au) = (u, A^* Au) = (u, u) = \|u\|^2.$$

Therefore $|\lambda| = 1$. □

Theorem 4.9.6

Theorem 4.9.6. Eigenvectors corresponding to distinct eigenvalues of self adjoint or unitary operator on a Hilbert space are orthogonal.

Proof. Let A be self adjoint and let u_1 and u_2 be eigenvectors corresponding to eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$. By Theorem 4.9.3, $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\lambda_1(u_1, u_2) = (\lambda_1 u_1, u_2) = (Au_1, u_2) = (u_1, A^* u_2) = (u_1, Au_2)$$

$$= (u_1, \lambda_2 u_2) = \overline{\lambda}_2(u_1, u_2) = \lambda_2(u_1, u_2).$$

Since $\lambda_1 \neq \lambda_2$, it must be that $(u_1, u_2) = 0$. □

Theorem 4.9.7

Theorem 4.9.7. For every eigenvalue λ of a bounded operator A, we have $|\lambda| \leq \|A\|$.

Proof. Suppose $Au = \lambda u$. We have $\|\lambda u\| = \|Au\|$ and so $|\lambda| \, \|u\| = \|Au\| \leq \|A\| \, \|u\|$. Therefore $|\lambda| \leq \|A\|$. □