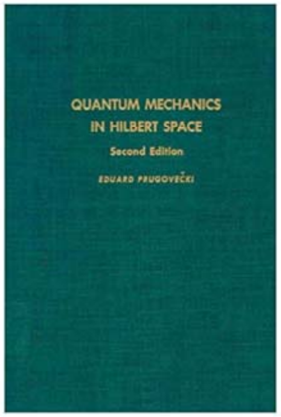
Modern Algebra

Chapter I. Basic Ideas of Hilbert Space Theory

I.1. Vector Spaces—Proofs of Theorems



Theorem I.1.1

Theorem I.1.1. Every vector space $\mathcal{V}$ has only one zero vector $\mathbf{0}$, and each element f of a vector space has one and only one additive inverse $(-f)$. For any $f \in \mathcal{V}$, we have $0f = \mathbf{0}$ and $(-1)f = (-f)$.

Proof. If $\mathbf{0}_1$ and $\mathbf{0}_2$ are both zero vectors, then by Axiom 3 of Definition I.1, $f = f + \mathbf{0}_1 = f + \mathbf{0}_2$ for all $f \in \mathcal{V}$. With $f = \mathbf{0}_1$ we have $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2$ and with $f = \mathbf{0}_2$ we have $\mathbf{0}_2 = \mathbf{0}_2 + \mathbf{0}_1$, so by Axiom 1, $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2 + \mathbf{0}_1 = \mathbf{0}_2$. Therefore the additive identity vector is unique.

Next,

$$\begin{aligned} f &= af \text{ by Axiom 7} \\ &= (1+0)f = 1f + 0f \text{ by Axiom 5} \\ &= f + 0f \text{ by Axiom 7} \end{aligned}$$

and so, by Axiom 3, $0f = \mathbf{0}$.

Theorem I.1.1 (continued)

Theorem I.1.1. Every vector space $\mathcal{V}$ has only one zero vector $\mathbf{0}$, and each element f of a vector space has one and only one additive inverse $(-f)$. For any $f \in \mathcal{V}$, we have $0f = \mathbf{0}$ and $(-1)f = (-f)$.

Proof (continued). We then have

$$\begin{aligned} (-1)f + f &= (-1)f + 1f \text{ by Axiom 7} \\ &= (-1+1)f \text{ by Axiom 5} \\ &= 0f = \mathbf{0}, \end{aligned}$$

and so f has an additive inverse and $(-f) = (-1)f$. Now suppose $f_1 \in \mathcal{V}$ is another additive inverse of f so that $f + f_1 = \mathbf{0}$. Then

$$\begin{aligned} (-f) &= (-f) + \mathbf{0} \text{ by Axiom 3} \\ &= (-f) + (f + f_1) = ((-f) + f) + f_1 \text{ by Axiom 2} \\ &= \mathbf{0} + f_1 = f_1 \text{ be Axioms 1 and 3.} \end{aligned}$$

That is, $(-f) = f_1$ and the additive inverse of f is unique. □

Theorem I.1.2

Theorem I.1.2. If the vector space $\mathcal{V}$ is n dimensional, where $n \in \mathbb{N}$, then there is at least one set $f_1, f_2, \ldots, f_n$ of linearly independent vectors, and each vector $f \in \mathcal{V}$ can be expanded as $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$, there the coefficients $a_1, a_2, \ldots, a_n$ are uniquely determined by f.

Proof. First, if $f = \mathbf{0}$ then we can just take $a_1 = a_2 = \cdots = a_n$ (by Theorem I.1.1 and Axiom 3). For $f \neq \mathbf{0}$, the equation $cf + c_1 f_2 + c_2 f_2 + \cdots c_n f_n = \mathbf{0}$ has a solution where $c \neq 0$ because $f_1, f_2, \ldots, f_n$ are linearly independent and $\mathcal{V}$ is dimension n (so $f, f_1, f_2, \ldots, f_n$ must be dependent, but if $c = 0$ then we would need $c_1 = c_2 = \cdots = c_n = 0$). So we get

$$f = \frac{-c_1}{c} f_1 + \frac{-c_2}{c} f_2 + \cdots + \frac{-c_n}{c} f_n,$$

and so scalars $a_1, a_2, \ldots, a_n$ exist as claimed.

Theorem I.1.2 (continued)

Theorem I.1.2. If the vector space $\mathcal{V}$ is n dimensional, where $n \in \mathbb{N}$, then there is at least one set $f_1, f_2, \dots, f_n$ of linearly independent vectors, and each vector $f \in \mathcal{V}$ can be expanded as $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$, there the coefficients $a_1, a_2, \dots, a_n$ are uniquely determined by f.

Proof (continued). If we also have $f = b_1 f_1 + b_2 f_2 + \cdots + b_n f_n$, then

$$\mathbf{0} = f - f = (a_1 f_1 + a_2 f_2 + \cdots + a_n f_n) - (b_1 f_1 + b_2 f_2 + \cdots + b_n f_n)$$

$$= (a_1 - b_1) f_1 + (a_2 - b_2) f_2 + \cdots + (a_n - b_n) f_n$$

by Axiom 1 and Axiom 5. But since $f_1, f_2, \dots, f_n$ are linearly independent then (by Definition I.1.2), $a_1 - b_1 = 0$, $a_2 - b_2 = 0$, ..., $a_n - b_n = 0$ and so $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$. That is, the choice of coefficients $a_1, a_2, \dots, a_n$ is unique, as claimed. □

Theorem I.1.3

Theorem I.1.3. If the set $\{g_1, g_2, \dots, g_n\}$ is a basis of n-dimensional vector space $\mathcal{V}$ (where $n \in \mathbb{N}$), then $m = n$. That is, all bases of an n-dimensional vector space are of the same size n.

Proof. Since $\mathcal{V}$ is n-dimensional, there are n linearly independent vectors $f_1, f_2, \dots, f_n$. Since $\{g_1, g_2, \dots, g_n\}$ is a basis, then

$$\begin{aligned} f_1 &= a_{11} g_1 + a_{21} g_2 + \cdots + a_{m1} g_m \\ f_2 &= a_{12} g_1 + a_{22} g_2 + \cdots + a_{m2} g_m \\ &\vdots \\ f_n &= a_{1n} g_1 + a_{2n} g_2 + \cdots + a_{mn} g_m. \end{aligned}$$

So if $x_1 f_1 + x_2 f_2 + \cdots x_n f_n = \mathbf{0}$ then substituting for $f_1, f_2, \dots, f_n$ we get

$$x_1 f_1 + x_2 f_2 + \cdots + x_n f_n = (a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n) g_1$$

$$+ (a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n) g_2 + \cdots + (a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n) g_m = \mathbf{0}.$$

Theorem I.1.3 (continued)

Proof (continued). Since $g_1, g_2, \dots, g_m$ are linearly independent, then $x_1, x_2, \dots, x_n$ satisfy the homogeneous system of equations

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0, \end{aligned}$$

and conversely any solution to this system of equations yields $x_1 f_1 + x_2 f_2 + \cdots + x_n f_n = \mathbf{0}$. But since $f_1, f_2, \dots, f_n$ are linearly independent then the only solution to the system of equations is the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. Finally, we have $m \le n$ since $\mathcal{V}$ is n-dimensional and $g_1, g_2, \dots, g_m$ are linearly independent (see Definition I.1.2). Since the homogeneous system of equations only has the trivial solution, then by Lemma I.1.A above, $n \le m$. Therefore $m = n$, as claimed. □

Theorem I.1.4

Theorem I.1.4. The Fundamental Theorem of Finite Dimensional Vector Spaces.
All complex (real) n-dimensional ($n \in \mathbb{N}$) vector spaces are isomorphic to the vector space $\mathbb{C}^n$ (or $\mathbb{R}^n$ in the case of real vector spaces).

Proof. Let $\mathcal{V}$ be an n-dimensional complex vector space. By Theorem I.1.2, there is a basis consisting of n vectors, $f_1, f_2, \dots, f_n$, and each given $f \in \mathcal{V}$ can be expanded as $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$ for unique $a_1, a_2, \dots, a_n \in \mathbb{C}$. So we define a mapping of $\mathcal{V}$ to $\mathbb{C}^n$ as $f \mapsto \alpha_f = [a_1, a_2, \dots, a_n]^T \in \mathbb{C}^n /$ Notice that this mapping is one to one (by the uniqueness of the a_i's) and onto. Now for $f, g \in \mathcal{V}$ with $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$ and $g = b_1 f_1 + b_2 f_2 + \cdots + a_n f_n$ we have

$$f + g = (a_1 + b_1) f_1 + (a_2 + b_2) f_2 + \cdots + (a_n + b_n) f_n \mapsto \cdots$$

Theorem I.1.4 (continued)

Proof (continued).

$$\cdots \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and for scalar $a \in \mathbb{C}$,

$$af = (aa_1)f_1 + (aa_2)f_2 + \cdots + (aa_n)f_n \mapsto \begin{bmatrix} aa_1 \\ aa_2 \\ \vdots \\ aa_n \end{bmatrix} = a \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

and so the mapping is an isomorphism. Hence, $\mathcal{V}$ is isomorphic to $\mathbb{C}^n$. Replacing field $\mathbb{C}$ with field $\mathbb{R}$, we see that real n-dimensional vector space is isomorphic to $\mathbb{R}$ and, more generally, n-dimensional vector space $\mathcal{V}$ over field F is isomorphic to F^n. □