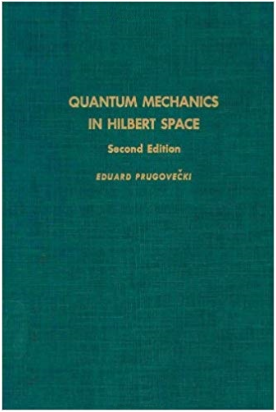
Modern Algebra

Chapter I. Basic Ideas of Hilbert Space Theory

I.2. Euclidean (pre-Hilbert) Spaces—Proofs of Theorems



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Theorem I.2.1

Theorem I.2.1. In a Euclidean space $\mathcal{E}$, the inner product $\langle f \mid g \rangle$ satisfies the relations:

(a) $\langle af \mid g \rangle = a^* \langle f \mid g \rangle$, and

(b) $\langle f + g \mid h \rangle = \langle f \mid h \rangle + \langle g \mid h \rangle$

for all $f, g, h \in \mathcal{E}$ and for every scalar a.

Proof. We have

$$\begin{aligned} \langle af \mid g \rangle &= \langle g \mid af \rangle^* \text{ by Definition I.2.1(2)} \\ &= (a\langle g \mid f \rangle)^* \text{ by Definition I.2.1(3)} \\ &= a^* \langle g \mid f \rangle^* \text{ since } (z_1 z_2)* = z_1^* z_2^* \\ &= a^* \langle f \mid g \rangle \text{ by Definition I.2.1(2),} \end{aligned}$$

and. . .

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Theorem I.2.1 (continued)

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Proof. ...

$$\begin{aligned}
\langle f + g \mid h\rangle &= \langle h \mid f + g\rangle^* \text{ by Definition I.2.1(2)} \\
&= (\langle h \mid f\rangle + \langle h \mid g\rangle)^* \text{ by Definition I.2.1(4)} \\
&= \langle h \mid f\rangle^* + \langle h \mid g\rangle^* \text{ since } (z_1 + z_2)* = z_1^* + z_2^* \\
&= \langle f \mid g\rangle + \langle g \mid h\rangle \text{ by Definition I.2.1(2).}
\end{aligned}$$

□

Theorem I.2.2

Theorem I.2.2. Schwarz-Cauchy Inequality.
Any two elements f, g of a Euclidean space $\mathcal{E}$ satisfies

$$|\langle f \mid g\rangle|^2 \leq \langle f \mid f\rangle\langle g \mid g\rangle.$$

Proof. For any $f, g \in \mathcal{E}$ and any $a \in \mathbb{C}$ we have $\langle f + ag \mid f + ag\rangle \geq 0$ by Definition I.2.1(1). If $\langle f \mid g\rangle = 0$ then the result holds (again, by Definition I.2.1(1)), so we can assume without loss of generality that $\langle f \mid g\rangle \neq 0$. Let $a = \lambda\langle f \mid g\rangle^*/|\langle f \mid g\rangle|$ where $\lambda \in \mathbb{R}$. Then $\langle f + ag \mid f + ag\rangle \geq 0$ implies

$$\begin{aligned} \langle f + ag \mid f + ag\rangle &= \langle f + ag \mid f\rangle_{\langle} f + ag \mid ag\rangle \text{ by Definition I.2.1(4)} \\ &= \langle f \mid f\rangle + \langle ag \mid f\rangle + \langle f \mid ag\rangle + \langle ag \mid ag\rangle \\ &\quad \text{by Theorem I.2.1(b)} \\ &= a^* a\langle g \mid g\rangle + \langle f \mid ag\rangle^* + \langle f \mid ag\rangle + \langle f \mid f\rangle \\ &\quad \text{by Definition I.2.1(2)} \ldots \end{aligned}$$

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\langle f + ag \mid f + ag\rangle &= \langle f + ag \mid f\rangle_{\langle} f + ag \mid ag\rangle \text{ by Definition I.2.1(4)} \\
&= \langle f \mid f\rangle + \langle ag \mid f\rangle + \langle f \mid ag\rangle + \langle ag \mid ag\rangle \\
&\quad \text{by Theorem I.2.1(b)} \\
&= a^*a\langle g \mid g\rangle + \langle f \mid ag\rangle^* + \langle f \mid ag\rangle + \langle f \mid f\rangle \\
&\quad \text{by Definition I.2.1(2)} \ldots
\end{aligned}$$

Theorem I.2.2 (continued 1)

Proof (continued). ...

$$\begin{aligned}
\langle f + ag \mid f + ag \rangle &= |a|^2 \langle g \mid g \rangle + a^* \langle f \mid g \rangle^* + a \langle f \mid g \rangle + \langle f \mid f \rangle \\
&\quad \text{by Definition I.2.1(3)} \\
&= \left| \lambda \frac{\langle f \mid g \rangle^*}{|\langle f \mid g \rangle|} \right|^2 \langle g \mid g \rangle + \lambda \frac{\langle f \mid g \rangle}{|\langle f \mid g \rangle|} \langle f \mid g \rangle^* \\
&\quad + \lambda \frac{\langle f \mid g \rangle^*}{|\langle f \mid g \rangle|} \langle f \mid g \rangle + \langle f \mid \rangle \\
&= \lambda^2 \langle g \mid g \rangle + 2\lambda |\langle f \mid g \rangle| + \langle f \mid \rangle \\
&\quad \text{since } |z^*| = |z| \text{ and } z^* z = |z|^2 \\
&\geq 0.
\end{aligned}$$

Define polynomial of real variable

$$p(\lambda) = \lambda^2 + \langle g \mid g \rangle + 2\lambda |\lambda f \mid g \rangle| + \langle f \mid f \rangle.$$

Theorem I.2.2 (continued 1)

Proof (continued). ...

$$\begin{array}{rcl} \langle f+ag \mid f+ag \rangle & = & |a|^2 \langle g \mid g \rangle + a^* \langle f \mid g \rangle^* + a \langle f \mid g \rangle + \langle f \mid f \rangle \\ & & \text{by Definition I.2.1(3)} \\ & = & \left| \lambda \dfrac{\langle f \mid g \rangle^*}{|\langle f \mid g \rangle|} \right|^2 \langle g \mid g \rangle + \lambda \dfrac{\langle f \mid g \rangle}{|\langle f \mid g \rangle|} \langle f \mid g \rangle^* \\ & & + \lambda \dfrac{\langle f \mid g \rangle^*}{|\langle f \mid g \rangle|} \langle f \mid g \rangle + \langle f \mid \rangle \\ & = & \lambda^2 \langle g \mid g \rangle + 2\lambda |\langle f \mid g \rangle| + \langle f \mid \rangle \\ & & \text{since } |z^*| = |z| \text{ and } z^* z = |z|^2 \\ & \geq & 0. \end{array}$$

Define polynomial of real variable

$$p(\lambda) = \lambda^2 + \langle g \mid g \rangle + 2\lambda |\lambda f \mid g \rangle| + \langle f \mid f \rangle.$$

Theorem I.2.2 (continued 2)

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Any two elements f, g of a Euclidean space $\mathcal{E}$ satisfies

$$|\langle f \mid g\rangle|^2 \leq \langle f \mid f\rangle\langle g \mid g\rangle.$$

Proof (continued). Then $p(\lambda)$ is a second degree nonnegative concave up polynomial and so it must have at most one root. This means the discriminant from the quadratic equation must no non-positive. So we need $(2|\langle f \mid g\rangle|)^2 - 4(\langle g \mid g\rangle)(\langle f \mid f\rangle) \leq 0$ or $\langle f \mid g\rangle|^2 \leq \langle f \mid f\rangle\langle g \mid g\rangle$, as claimed. $\square$

Theorem I.2.3

Theorem I.2.3. In a Euclidean space $\mathcal{E}$ with inner product $\langle f \mid g \rangle$, the real-valued function $\|f\| = \sqrt{\langle f \mid f \rangle}$ is a norm.

Proof. We check the four parts of Definition I.2.2.

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(1) If $f \neq \mathbf{0}$ then $\langle f \mid f \rangle > 0$ by Definition I.2.1(1), so $\|f\| = \sqrt{\langle f \mid f \rangle} > 0$ for $f \neq \mathbf{0}$.

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(3)

$$\begin{aligned} \|af\| &= \sqrt{\langle af \mid af \rangle} = \sqrt{a^* a \langle f \mid f \rangle} \text{ by Definition I.2.1(3)} \\ &\quad \text{and Theorem I.2.1(a)} \\ &= \sqrt{a^* a}\sqrt{\langle f \mid f \rangle} = |a|\|f\| \text{ since } |z|^2 = z^* z. \end{aligned}$$

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Theorem I.2.3 (continued)

Proof (continued). (4) For the Triangle Inequality, we have

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g \mid f+g\rangle \\
&= \langle f \mid f\rangle + \langle f \mid g\rangle + \langle g \mid f\rangle + \langle g \mid g\rangle \\
&\quad \text{by Definition I.2.1(4) and Theorem I.2.1(b)} \\
&= \|f\|^2 + \langle f \mid g\rangle + \langle f \mid g\rangle^* + \langle g \mid g\rangle \text{ by Definition I.2.1(2)} \\
&= \|f\|^2 + 2\mathrm{Re}(\langle f \mid g\rangle) + \|g\|^2 \text{ since } z + z^* = 2\mathrm{Re}(z) \\
&\leq \|f\|^2 + r|\mathrm{Re}(\langle f \mid g\rangle)| + \|g\|^2 \\
&\leq \|f\|^2 + 2|\langle f \mid g\rangle| + \|g\|^2 \text{ since } |\mathrm{Re}(z)| \leq |z| \\
&\leq \|f\|^2 + 2\sqrt{\langle f \mid f\rangle\langle g \mid g\rangle} + \|g\|^2 \\
&\quad \text{by Schwarz-Cauchy Inequality (Theorem I.2.2)} \\
&= \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2,
\end{aligned}$$

and so $\|f+g\| \leq \|f\| + \|g\|$. □

Theorem I.2.3 (continued)

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&\le \|f\|^2 + 2\sqrt{\langle f \mid f\rangle\langle g \mid g\rangle} + \|g\|^2 \\
&\qquad \text{by Schwarz-Cauchy Inequality (Theorem I.2.2)} \\
&= \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\|+\|g\|)^2,
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and so $\|f+g\| \le \|f\| + \|g\|$. □

Theorem I.2.4

Theorem I.2.4. If S is a finite of countably infinite set of vectors in a Euclidean space $\mathcal{E}$ and $\mathcal{V}$ is the vector subspace of $\mathcal{E}$ spanned by S, then there is an orthonormal system T of vectors which spans $\mathcal{V}$; that is, for which $\text{span}(T) = \mathcal{V}$ (that is, the set of all linear combinations of elements of T; Prugovečki denotes the space of T as (T)). T is a finite set when S is a finite set.

Proof. Let $S = \{f_1 f_2, \ldots\}$. Define g_1 to be the first nonzero vector in S. Then recursively define g_n for $n \geq 2$ as $g_n = f_m$ where m is the minimum index such that $\{g_1, g_2, \ldots, g_{n-1}, f_n\}$ is linearly independent; if no such m exists, take $S_0 = \{g_1, g_2, \ldots, g_{n-1}\}$ (so if S_0 is finite then Tfinite). Then $S_0 = \{g_1, g_2, \ldots\}$. Notice that S_0 is linearly independent and $\text{span}(S_0) = \text{span}(S)$. We now create orthonormal set T from S_0 using the Gram-Schmidt process (or the "Schmidt orthonormalization procedure").

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Theorem I.2.4 (continued 1)

Proof (continued). Since $g \neq \mathbf{0}$, let $e_1 = g_1/\|g_1\|$. Then recursively define e_n for $n \geq 2$ in terms of g_n and $e_1, e_2, \ldots, e_{n-1}$ as

$$e_n = \frac{g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \langle e_{n-2} \mid g_n \rangle e_{n-2} - \cdots - \langle e_1 \mid g_n \rangle e_1}{\|g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \langle e_{n-2} \mid g_n \rangle e_{n-2} - \cdots - \langle e_1 \mid g_n \rangle e_1\|}.$$

We now show the denominator here is nonzero. ASSUME

$$g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \langle e_{n-2} \mid g_n \rangle e_{n-2} - \cdots - \langle e_1 \mid g_n \rangle e_1 = \mathbf{0}.$$

We have such g_n expanded as a linear combination of $e_1, e_2, \ldots, e_k$ (so $\mathrm{span}(g_1, g_2, \ldots, g_k) \subset \mathrm{span}(e_1, e_2, \ldots, e_k)$) and so we can solve for g_k as follows (where $c_{\ell,}$ are scalars):

$$\begin{aligned} g_1 &= c_{11} e_1 \\ g_2 &= c_{21} e_1 + c_{22} e_2 \\ &\vdots \\ g_{n-1} &= c_{n-1,1} e_1 + c_{n-1,2} e_2 + \cdots + c_{n-1,n-1} e_{n-1}. \end{aligned}$$

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Theorem I.2.4 (continued 2)

Proof (continued). We can inductively solve this system of equations for e_1 in terms of g_1, solve for e_2 in terms of g_1 and g_2, ..., solve for e_{n-1} in terms of $g_1, g_2, \ldots, g_{n-1}$ (this is sometimes called "back substitution"). This implies $\text{span}(e_1, e_2, \ldots, e_k) \subset \text{span}(g_1, g_2, \ldots, g_k)$ and hence $\text{span}(e_1, e_2, \ldots, e_k) = \text{span}(g_1, g_2, \ldots, g_k)$ and $\text{span}(T) = \text{span}(S_0) = \text{span}(S)$. If the assumed equality holds then g_n is a linear combination of $e_1, e_2, \ldots, e_{n-1}$, but then we can substitute for e_i its expression in terms of $g_1, g_2, \ldots, g_i$ (for $1 \leq i \leq n-1$) and then we can express g_n as a linear combination of $g_1, g_2, \ldots, g_{n-1}$, CONTRADICTING the choice of g_n. So the denominator in the formula for e_n is not 0 and e_n is defined.

The vectors in $I = \{e_1, e_2, \ldots\}$ are unit vectors (i.e., normalized). We now show orthogonality. Let $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$. We give an induction argument.

Theorem I.2.4 (continued 2)

Proof (continued). We can inductively solve this system of equations for e_1 in terms of g_1, solve for e_2 in terms of g_1 and g_2, ..., solve for e_{n-1} in terms of $g_1, g_2, \ldots, g_{n-1}$ (this is sometimes called "back substitution"). This implies $\text{span}(e_1, e_2, \ldots, e_k) \subset \text{span}(g_1, g_2, \ldots, g_k)$ and hence $\text{span}(e_1, e_2, \ldots, e_k) = \text{span}(g_1, g_2, \ldots, g_k)$ and $\text{span}(T) = \text{span}(S_0) = \text{span}(S)$. If the assumed equality holds then g_n is a linear combination of $e_1, e_2, \ldots, e_{n-1}$, but then we can substitute for e_i its expression in terms of $g_1, g_2, \ldots, g_i$ (for $1 \le i \le n-1$) and then we can express g_n as a linear combination of $g_1, g_2, \ldots, g_{n-1}$, CONTRADICTING the choice of g_n. So the denominator in the formula for e_n is not 0 and e_n is defined.

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Theorem I.2.4 (continued 3)

Proof (continued). Suppose we have shown that $\langle e_i \mid e_j \rangle = \delta_{ij}$ for $i, j = 1, 2, \ldots, n-1$. Then for $m < n$

$$\begin{aligned}
\langle e_m \mid e_n \rangle &= \left\langle e_m \;\middle|\; \frac{g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \cdots - \langle e_1 \mid g_n \rangle e_1}{\| g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \cdots - \langle e_1 \mid g_n \rangle e_1 \|} \right\rangle \\
&= \frac{1}{\| g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \cdots - \langle e_1 \mid g_n \rangle e_1 \|} \times \\
&\quad \left(\langle e_m \mid g_n \rangle - \sum_{k=1}^{n-1} \langle e_k \mid g_n \rangle \langle e_m \mid e_k \rangle \right) \\
&= \frac{1}{\| g_n - \langle e_{n-1} \mid g_n \rangle e_{n-1} - \cdots - \langle e_1 \mid g_n \rangle e_1 \|} \times \\
&\quad \left(\langle e_m \mid g_n \rangle - \sum_{k=1}^{n-1} \langle e_k \mid g_n \rangle \delta_{mk} \right) = 0.
\end{aligned}$$

Theorem I.2.4 (continued 4)

Theorem I.2.4. If S is a finite of countably infinite set of vectors in a Euclidean space $\mathcal{E}$ and $\mathcal{V}$ is the vector subspace of $\mathcal{E}$ spanned by S, then there is an orthonormal system T of vectors which spans $\mathcal{V}$; that is, for which $\mathrm{span}(T) = \mathcal{V}$ (that is, the set of all linear combinations of elements of T; Prugovečki denotes the space of T as (T)). T is a finite set when S is a finite set.

Proof (continued). So $\langle e_i \mid e_j \rangle = \delta_{ij}$ for $i, j = 1, 2, \ldots, n-1, n$ and by induction T is orthonormal. □

Theorem I.2.5

Theorem I.2.5. All complex Euclidean n-dimensional spaces are isomorphic to $\ell^2(n)$ and consequently mutually isomorphic.

Proof. If $\mathcal{E}$ is an n-dimensional Euclidean space then there is, by Theorem I.1.2, a set of n vectors $f_1, f_2, \ldots f_n$ spanning $\mathcal{E}$. By Theorem I.2.4, there is an orthonormal system of n vectors $e_1, e_2, \ldots, e_n$ which also spaces $\mathcal{E}$. Consider the mapping $f \mapsto [\langle e_1 \mid r\rangle, \langle e_2 \mid f\rangle, \ldots, \langle e_n \mid f\rangle]^T$. It is to be shown that this mapping is an isomorphism between $\mathcal{E}$ and $\ell^2(n)$ in Exercise I.2.7. $\square$

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