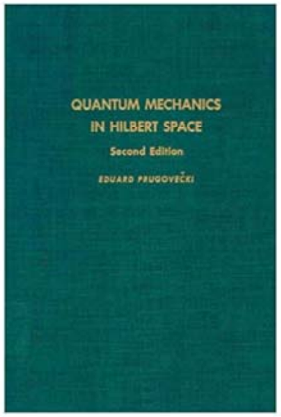
Modern Algebra

Chapter I. Basic Ideas of Hilbert Space Theory

I.3. Metric Spaces—Proofs of Theorems



Theorem I.3.1

Theorem I.3.1. If a sequence $\xi_1, \xi_2, \ldots,$ in a metric space $\mathcal{M}$ converges to some $\xi \in \mathcal{M}$ then its limit is unique, and the sequence is a Cauchy sequence.

Proof. If $\{\xi_1, \xi_2, \ldots\}$ converges to $\xi \in \mathcal{M}$ and to $\eta \in \mathcal{M}$ then for any $\varepsilon > 0$ there exists positive $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that $d(\xi, \xi_n) < \varepsilon/2$ for $n > N_1(\varepsilon)$ and $d(\eta, \xi_n) < \varepsilon/2$ for $n > N_2(\varepsilon)$. Consequently, for $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ we have by the Triangle Inequality that

$$d(\xi, \eta) \le d(\xi, \xi_n) + d(\xi_n, \eta) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $d(\xi, \eta) = 0$ and so (by Definition 3.1(1) and (2)), we have $\xi = \eta$ and so the limit of the sequence is unique.

If $m, n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ then by the Triangle Inequality

$$d(\xi_m, \xi_n) \le d(\xi_m, \xi) + d(\xi, \xi_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and so the convergent sequence is Cauchy. □

Theorem I.3.2

Theorem I.3.2. Every incomplete metric space $\mathcal{M}$ can be embedded in a complete metric space $\tilde{\mathcal{M}}$, called the *completion* of $\mathcal{M}$.

Proof. Let $\{\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \ldots\}$ be a Cauchy sequence in $\tilde{\mathcal{M}}$, where $\tilde{\xi}^{(k)}$ is the equivalence class containing the Cauchy sequence $\tilde{\xi}^{(k)} = \{\xi_1^{(k)}, \xi_2^{(k)}, \ldots\}$ of elements of $\mathcal{M}$. We will construct a Cauchy sequence $\tilde{\eta} \in \tilde{\mathcal{M}}$ which is the limit of this given Cauchy sequence of equivalence classes $\{\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \ldots\} \in \tilde{\mathcal{M}}$. For each $k \in \mathbb{N}$, choose element $\xi_n^{(k)} \in \mathcal{M}$ such that $d(\xi_m^{(k)}, \xi_n^{(k)}) < 1/k$ for all $m > N_k$ for some positive $N(1/k)$ (which can be done since $\{\xi_1^{(k)}, \xi_2^{(k)}, \ldots\}$ is a Cauchy sequence); denote $\eta_k = \xi_n^{(k)}$ so that $d(\xi_m^{(k)}, \eta_k) < 1/k$ for $m > N(1/k)$. Next, consider $\tilde{\eta}_k = \{\eta_k, \eta_k, \ldots\}$ and $\tilde{\xi}_m^{(k)} = \{\xi_m^{(k)}, \xi_m^{(k)}, \ldots\}$ in $\tilde{\mathcal{M}}_S$. By choice of η_k and $\xi_m^{(k)}$ we have

$$d_S(\tilde{\xi}_m^{(k)}, \tilde{\eta}_k) = d(\xi_m^{(k)}, \eta_k) < 1/k.$$

Theorem I.3.2 (continued 1)

Proof (continued). By Exercise I.3.6 (with constant sequence $\tilde{\eta}_k = \{\xi_k, \xi_k, \ldots\}$ of Exercise I.3.6 replaced with constant sequence $\tilde{\eta}_m^{(k)} = \{\xi_m^{(k)}, \xi_m^{(k)}, \ldots\}$ here, and sequence $\tilde{\xi} = \{\xi_1, \xi_2, \ldots\}$ of Exercise I.3.6 replaced with sequence $\tilde{\xi}^{(k)} = \{\xi_1^{(k)}, \xi_2^{(k)}, \ldots\}$ here),

$$\lim_{m\to\infty} d_S(\tilde{\xi}_m^{(k)}, \tilde{\xi}^{(k)}) = 0.$$

Then by Exercise I.3.5 (with arbitrary sequence η of exercise I.3.5 replaced with constant sequence $\tilde{\eta}_k$ here, and sequence $\{\xi_1, \xi_2, \ldots\}$ convergent to ξ in Exercise I.3.5 replaced with sequence $\{\tilde{\xi}_1^{(k)}, \tilde{\xi}_2^{(k)}, \ldots\}$, which we just argued is convergent to $\tilde{\xi}^{(k)}$, here),

$$\lim_{m\to\infty} d_S(\tilde{\xi}_m^{(k)}, \tilde{\eta}_k) = d_S(\tilde{\xi}^{(k)}, \tilde{\eta}_k).$$

Consequently,

$$d_S(\tilde{\xi}^{(k)}, \tilde{\eta}_k) = \lim_{m\to\infty} d_S(\tilde{\xi}_m^{(k)}, \tilde{\eta}_k) \le 1/k. \quad (*)$$

Theorem I.3.2 (continued 2)

Proof (continued). Now we have chosen an η_k such that $\tilde{\eta}_k = \{\eta_k, \eta_k, \ldots\}$ satisfies $(*)$ for each $k \in \mathbb{N}$. So this gives the sequence $\tilde{\eta} = \{\eta_1, \eta_2, \ldots\}$ which we now show is Cauchy and then show is the limit of the given Cauchy sequence $\{\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \ldots\} \in \tilde{\mathcal{M}}$. We have

$$\begin{aligned} d(\eta_m, \eta_n) &= d_S(\tilde{\eta}_m \tilde{\eta}_n), \text{ where } \tilde{\eta}_m \text{ and } \tilde{\eta}_n \text{ are constant} \\ &\quad \text{sequences } \tilde{\eta}_m = \{\eta_m, \eta_m, \ldots\} \text{ and } \tilde{\eta}_n = \{\eta_n, \eta_n, \ldots\}; \\ &\quad \text{equality follows from the definition of } d_S \text{ on } \tilde{\mathcal{M}}_S \\ &\leq d_S(\tilde{\eta}_m, \tilde{\xi}^{(m)}) + d_S(\tilde{\xi}^{(m)}, \tilde{\xi}^{(n)}) + d_S(\tilde{\xi}^{(n)}, \tilde{\eta}_m) \text{ by two} \\ &\quad \text{applications of the Triangle Inequality for } d_S, \text{ since } d_S \\ &\quad \text{is a metric on } \tilde{\mathcal{M}}_S \text{ and each sequence here is Cauchy;} \\ &\quad \text{notice that } \tilde{\xi}^{(m)}, \tilde{\xi}^{(n)} \text{ are Cauchy sequences here,} \\ &\quad \text{not equivalence classes} \\ &\leq 1/m + d_S(\tilde{\xi}^{(m)}, \tilde{\xi}^{(n)}) + 1/n \text{ by } (*). \end{aligned}$$

Theorem I.3.2 (continued 3)

Proof (continued). Since $\{\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \ldots\}$ is a Cauchy sequence of equivalence classes by hypothesis, then by making m and n sufficiently large, we can make $d(\eta_m, \eta_n)$ arbitrarily small (i.e., less than ε for any given $\varepsilon > 0$). So sequence $\tilde{\eta} = \{\eta_1, \eta_2, \ldots\}$ is Cauchy and $\tilde{\eta} \in \tilde{\mathcal{M}}_S$. Now let $\tilde{\eta}$ also denote the equivalence class containing Cauchy sequence $\{\eta_1, \eta_2, \ldots\}$, so $\tilde{\eta} \in \tilde{\mathcal{M}}$. We have by the Triangle Inequality for metric d_E on $\tilde{\mathcal{M}}$,

$$d_E(\tilde{\eta}, \tilde{\xi}^{(k)}) \leq d_E(\tilde{\eta}, \tilde{\eta}_k) + d_E(\tilde{\eta}_k, \tilde{\xi}^{(k)}) \quad (**)$$

where $\tilde{\eta}_k$ is the equivalence class in $\tilde{\mathcal{M}}$ containing Cauchy sequence $\{\eta_k, \eta_k, \ldots\}$. Now $d_S(\tilde{\xi}^{(k)}, \tilde{\eta}_k) \leq 1/k$ by $(*)$ and so $d_E(\tilde{\xi}^{(k)}, \tilde{\eta})k) \leq 1/k$ (with the appropriate caution concerning equivalence classes and sequences). By Exercise I.3.6 (with sequences $\tilde{\xi}$ and $\tilde{\xi}_k$ of Exercise I.3.6 replaced with sequences $\tilde{\eta}$ and $\tilde{\eta}_k$, respectively, here) we have $\lim_{k\to\infty} d_S(\tilde{\eta}, \tilde{\eta}_k) = 0$.

Theorem I.3.2 (continued 4)

Proof (continued). Since $d_E(\tilde{\eta}, \tilde{\eta}_k) = d_S(\tilde{\eta}, \tilde{\eta}_k)$ (again, with sequence/equivalence class caution), then $\lim_{k\to\infty} d_E(\tilde{\eta}, \tilde{\eta}_k) = 0$. So, for k sufficiently large, we see from $(**)$ that $d_E(\tilde{\eta}, \tilde{\xi}^{(k)})$ can be made arbitrarily small. Therefore, $\lim_{k\to\infty} \tilde{\xi}^{(k)} = \tilde{\eta}$ in $\tilde{\mathcal{M}}$ and so arbitrary sequence $\{\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \ldots\}$ in $\tilde{\mathcal{M}}$ converges in $\tilde{\mathcal{M}}$. Hence, $\tilde{\mathcal{M}}$ is a compete metric space.

Finally, we show that we can embed $\mathcal{M}$ in $\tilde{\mathcal{M}}$. Let $\xi \in \mathcal{M}$. Map $\xi \in \mathcal{M}$ to the equivalence class $\tilde{\xi} \in \tilde{\mathcal{M}}$ containing the Cauchy sequence $\{\xi, \xi, \ldots\}$. This mapping is "clearly" one to one and isometric (since $d(\xi, \eta) = d_S(\tilde{\xi}, \tilde{\eta}) = d_E(\tilde{\xi}, \tilde{\eta})$). Let $\mathcal{M}'$ be the image of $\mathcal{M}$ under this mapping. If $\tilde{\eta} \in \tilde{\mathcal{M}}$ where equivalence class $\tilde{\eta}$ contains Cauchy sequence $\{\eta_1, \eta_2, \ldots\} \in \tilde{\mathcal{M}}_S$.

Theorem I.3.2 (continued 5)

Theorem I.3.2. Every incomplete metric space $\mathcal{M}$ can be embedded in a complete metric space $\tilde{\mathcal{M}}$, called the *completion* of $\mathcal{M}$.

Proof (continued). Then for given $\varepsilon > 0$, by Exercise I.3.6 (with sequences $\tilde{\xi}$ and $\tilde{\xi}_k$ of Exercise I.3.6 replaced with sequences $\tilde{\eta} = \{\eta_1, \eta_2, \ldots\}$ and $\tilde{\eta}_k = \{\eta_k, \eta_k, \ldots\}$ here), there is positive $N(\varepsilon)$ such that for all $k > N(\varepsilon)$ we have $d_S(\tilde{\eta}, \tilde{\eta}_k) < \varepsilon$. Since $d_S(\tilde{\eta}, \tilde{\eta}_k) = d_E(\tilde{\eta}, \tilde{\eta}_k)$ (caution!), this shows that $\mathcal{M}$ is everywhere dense in $\tilde{\mathcal{M}}$. That is, metric space $\mathcal{M}$ is densely embedded in complete metric space $\mathcal{M}$, as claimed. □