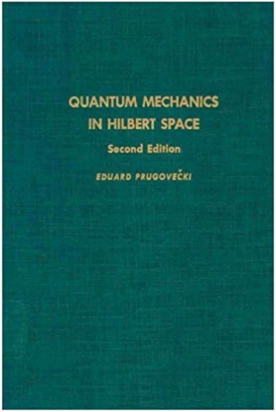
Modern Algebra

Chapter I. Basic Ideas of Hilbert Space Theory

I.4. Hilbert Spaces—Proofs of Theorems



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Proof (continued). ... by the definition of "equivalence" on $\tilde{\mathcal{E}}_S$, and then

$$\lim_{n\to\infty} \|af_n' - af_n''\| = |a| \lim_{n\to\infty} \|f_n' - f_n''\| = 0.$$

So $\tilde{f}' + \tilde{g} \sim \tilde{f}'' + \tilde{g}$ and $a\tilde{f}' \sim a\tilde{f}''$. So we can define vector addition and scalar multiplication on $\tilde{\mathcal{E}}$ using representatives of equivalence classes and the resulting definition is well-defined (i.e., independent of representatives used). This then gives $\tilde{\mathcal{E}}$ a vector space structure.

Next, we define the complex function on $\tilde{\mathcal{E}}_S \times \tilde{\mathcal{E}}$ of $\langle \tilde{f} \mid \tilde{g}\rangle_S = \lim_{n\to\infty}\langle f_n \mid g_n\rangle$. But we need to confirm that the limit here actually exists. First, we have the inequality

$$\begin{aligned} |\langle f_m \mid g_m\rangle - \langle f_n \mid g_n\rangle| &= |\langle f_m - f_n \mid g_m\rangle + \langle f_n \mid g_m - g_n\rangle| \\ &\quad \text{by Definition I.2.1(4) and Theorem I.2.1(b)} \\ &\leq |\langle f_m - f_n \mid g_m\rangle| + |\langle f_n \mid g_m - g_n\rangle| \\ &\leq \|f_m - f_n\|\|g_m\| + \|f_n\|\|g_m - g_n\| \\ &\quad \text{by the Schwarz-Cauchy Inequality (Thm I.2.2).} \end{aligned}$$

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Theorem I.4.1 (continued 2)

Proof (continued). Now a Cauchy sequence is bounded (let $\varepsilon > 0$, then there is positive $N(\varepsilon)$ such that for all $m, n > N(\varepsilon)$ we have $\|f_n - f_m\| < \varepsilon$ and so for a fixed $m' > N(\varepsilon)$ and for all $n > N(\varepsilon)$ we have $\|f_n\| - \|f_{m'}\| \le \|f_n - f_{m'}\| < \varepsilon$ or $\|f_n\| < \|f_{m'}\| + \varepsilon$ and then the sequence is bounded by $\max\{\|f_1\|, \|f_2\|, \ldots, \|f_{m'-1}\|, \|f_{m'}\| + \varepsilon\}$), so the above inequality implies that $|\langle f_m \mid g_m\rangle = \langle f_n \mid g_n\rangle|$ can be made arbitrarily small by making m and n sufficiently large, since $\|f_m - f_n\| \to 0$ and $\|g_m - g_n\| \to 0$ as $m, n \to \infty$ since $\{f_1, f_2, \ldots\}$ and $\{g_1, g_2, \ldots\}$ are Cauchy. Therefore the sequence of complex numbers $\{\langle f_1 \mid g_1\rangle, \langle f_2 \mid g_2\rangle, \ldots\}$ is a Cauchy sequence and since $\mathbb{C}$ is complete then the sequence converges and $\langle \tilde{f} \mid \tilde{g}\rangle_S = \lim_{n\to\infty}\langle f_n \mid g_n\rangle$ exists.

If $\tilde{f}' \sim \tilde{f}''$ are elements of $\tilde{\mathcal{E}}_S$ then we have from the inequality

$$\begin{aligned} |\langle f_n' \mid g_n\rangle - \langle f_n'' \mid g_n\rangle| &= |\langle f_n' - f_n'' \mid g_n\rangle| \text{ by Theorem I.2.1(b)} \\ &\le \|f_n' - f_n''\|\|g_n\| \end{aligned}$$

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Proof (continued). we have $\lim_{n\to\infty} \|f'_n - f''_n\| = d(f'_n, f''_n) = 0$ by the definition of the equivalence relation on $\tilde{\mathcal{E}}_S$, and so $\lim_{n\to\infty} |\langle f'_n, g_n\rangle - \langle f''_n \mid g_n\rangle| = 0$ (again, the fact that $\{g_1, g_2, \ldots\}$ is Cauchy implies $\|g_n\|$ is bounded) and so $\langle f'_n \mid g_n\rangle = \langle f''_n \mid g_n\rangle$. So $\langle \tilde{f} \mid \tilde{g}\rangle_S$ can be used to define an inner product on the equivalence classes of $\tilde{\mathcal{E}}_S$; that is, we can define $\langle \tilde{f} \mid \tilde{g}\rangle$ on $\tilde{\mathcal{E}} \times \tilde{\mathcal{E}}$ where $\tilde{f}, \tilde{g} \in \tilde{\mathcal{E}}$ are equivalence classes and we define $\langle \tilde{f} \mid \tilde{g}\rangle = \langle \tilde{f} \mid \tilde{g}\rangle_S$ where on the right hand side $\tilde{f}$ and $\tilde{g}$ are Cauchy sequences (representatives) of the equivalence classes $\tilde{f}$ and $\tilde{g}$, respectively, on the left hand side. By Exercise I.4.4, $\langle \cdot \mid \cdot\rangle$ defines an inner product on $\tilde{\mathcal{E}}$ (that is, $\langle \cdot \mid \cdot\rangle$ satisfies the four parts of Defn I.2.1).

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Theorem I.4.2

Theorem I.4.2. Every subspace of a separable Euclidean space is a separable Euclidean space.

Proof. Let $\mathcal{E}_1$ be a (vector) subspace of Euclidean space $\mathcal{E}$. Then $\mathcal{E}_1$ itself is a Euclidean space by Exercise I.2.6. We now construct a countable dense subset $S = \{g_{11}, g_{12}, g_{22}, g_{13}, g_{23}, \ldots\}$ of $\mathcal{E}_1$.

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Since $\mathcal{E}$ is separable, there is a dense subset $R = \{f_1, f_2, \ldots\}$ of $\mathcal{E}$. For $m, n \in \mathbb{N}$, if there is an element of $\mathcal{E}_1$ within a distance $1/m$ of f_n, then denote is as g_{mn} (so that $\|g_{mn} - f_n\| < 1/m$); if no such element of $\mathcal{E}_1$ exists, then take $g_{mn} = \mathbf{0}$. Then set $S = \{g_{11}, g_{12}, g_{22}, g_{13}, g_{23}, \ldots\}$ is countable. Let $h \in \mathcal{E}_1$ be given and let $m \in \mathbb{N}$ be arbitrary. Since R is dense in $\mathcal{E}$ then there is $f_n \in R$ such that $\|h - f_n\| < 1/m$.

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Theorem I.4.2 (continued)

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Proof (continued). Since $h \in \mathcal{E}_1$ and $\|h - f_n\| < 1/m$ then $g_{mn} \neq \mathbf{0}$ and we have

$$\begin{aligned} \|h - g_{mn}\| &= \|h - f_n + f_n - g_{mn}\| \leq \|h - f_n\| + \|f_n - g_{mn}\| \\ &< 1/m + 1/m \text{ by the choice of } g_{mn} \\ &= 1/(2m). \end{aligned}$$

For $\varepsilon > 0$ given, choose $m > 1/(2\varepsilon)$ and then we see that S is dense in $\mathcal{E}_1$ so that $\mathcal{E}_1$ is separable. $\square$

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Theorem I.4.3

Theorem I.4.3. The set $\ell^2(\infty)$ of all one-column complex matrices α with countable number of elements, $\alpha = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$ for which $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ becomes a separable Hilbert space, also denoted $\ell^2(\infty)$, if the vector operations are defined by

$$\alpha+\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \end{bmatrix}, \text{ and } a\alpha = a \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} aa_1 \\ aa_2 \\ \vdots \end{bmatrix}$$

for any scalar $a \in \mathbb{C}$, and the inner product is defined by $\langle \alpha \mid \beta \rangle = \sum_{k=1}^{\infty} a_k^* b_k$.

Theorem I.4.3 (continued 1)

Proof (continued). First, we establish that $\ell^2(\infty)$ is actually a vector space. To do so, we need to confirm that $\ell^2(\infty)$ is closed under vector addition and scalar multiplication (each of the seven axioms in Definition I.1.1 then clearly hold). For $\alpha, \beta \in \ell^2(\infty)$ as described above, we consider for each $v \in \mathbb{N}$ $[a_1, a_2, \ldots, a_v]^T, [b_1, b_2, \ldots, b_v]^T \in \ell^2(\infty)$, so that by the Triangle Inequality on $\ell^2(v)$:

$$\left\{\sum_{k=1}^{v} |a_k + b_k|^2\right\}^{1/2} \leq \left\{\sum_{k=1}^{v} |a_k|^2\right\}^{1/2} + \left\{\sum_{k=1}^{v} |b_k|^2\right\}^{1/2}.$$

Then with $v \to \infty$, we get $\sum_{k=1}^{\infty} |a_k + b_k|^2 < \infty$ since $\alpha, \beta \in \ell^2(\infty)$, and so $\alpha + \beta \in \ell^2(\infty)$. Next, for $a \in \mathbb{C}$ we have $\sum_{k=1}^{\infty} |aa_k|^2 = \sum_{k=1}^{\infty} |a|^2 |a_k|^2 = |a|^2 \sum_{k=1}^{\infty} |a_k|^2 < \infty$ and so $a\alpha \in \ell^2(\infty)$. Therefore, $\ell^2(\infty)$ is a vector space.

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Then with $v \to \infty$, we get $\sum_{k=1}^{\infty} |a_k + b_k|^2 < \infty$ since $\alpha, \beta \in \ell^2(\infty)$, and so $\alpha + \beta \in \ell^2(\infty)$. Next, for $a \in \mathbb{C}$ we have $\sum_{k=1}^{\infty} |aa_k|^2 = \sum_{k=1}^{\infty} |a|^2 |a_k|^2 = |a|^2 \sum_{k=1}^{\infty} |a_k|^2 < \infty$ and so $a\alpha \in \ell^2(\infty)$. Therefore, $\ell^2(\infty)$ is a vector space.

Theorem I.4.3 (continued 2)

Proof (continued). In order to show $\ell^2(\infty)$ is a Euclidean space, we must first show that $\langle \alpha \mid \beta \rangle = \sum_{k=1}^{\infty} a_k^* b_k$ is actually a complex number (that is, the series converges). As above, for $\alpha, \beta \in \ell^2(\infty)$ we consider $[a_1, a_2, \ldots, a_v]^T, [b_1, b_2, \ldots, b_n]^T \in \ell^2(\infty)$ and by the Schwarz-Cauchy Inequality for $\ell^2(v)$ (Theorem I.2.2),

$$\sum_{k=1}^{v} |a_k^* b_k| \leq \left\{ \sum_{k=1}^{v} |a_k|^2 \right\}^{1/2} \left\{ \sum_{k=1}^{v} |b_k|^2 \right\}^{1/2}$$

for all $v \in \mathbb{N}$. Letting $v \to \infty$ we have

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since $\alpha, \beta \in \ell^2(\infty)$.

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since $\alpha, \beta \in \ell^2(\infty)$.

Theorem I.4.3 (continued 3)

Proof (continued). So $\sum_{k=1}^{\infty} a_k^* b_k$ is an absolutely convergent series and, since $\mathbb{C}$ is complete, then the series is convergent (see my online Complex Analysis 1 [MATH 5510] notes a `http://faculty.etsu.edu/gardnerr/5510/notes/III-1.pdf`; see Proposition III.1.1); that is, $\langle \alpha \mid \beta \rangle$ is defined.

To complete the proof that $\ell^2(\infty)$ is a Euclidean space, we now need to confirm that $\langle \alpha \mid \beta \rangle$ satisfies the four properties of Definition I.2.1, which is to be done in Exercise I.4.6.

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Next, we prove $\ell^2(\infty)$ is complete. Let $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ be a Cauchy sequence in $\ell^2(\infty)$ where $\alpha^{(n)} = [a_1^{(n)}, a_2^{(n)}, \ldots]^T$. For any $k \in \mathbb{N}$ we have

$$|a_k^{(m)} - a_k^{(n)}| = \sqrt{|a_k^{(m)} - a_k^{(n)}|^2} \leq \sqrt{\sum_{k=1}^{\infty} |a_k^{(m)} - a_k^{(n)}|^2} = \|\alpha^{(m)} - \alpha^{(n)}\|,$$

and since $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ is a Cauchy sequence then $\|\alpha^{(m)} - \alpha^{(n)}\|$ can be made arbitrarily small by making m and n sufficiently large.

Theorem I.4.3 (continued 3)

Proof (continued). So $\sum_{k=1}^{\infty} a_k^* b_k$ is an absolutely convergent series and, since $\mathbb{C}$ is complete, then the series is convergent (see my online Complex Analysis 1 [MATH 5510] notes a `http://faculty.etsu.edu/gardnerr/5510/notes/III-1.pdf`; see Proposition III.1.1); that is, $\langle \alpha \mid \beta \rangle$ is defined.

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Theorem I.4.3 (continued 4)

Proof (continued). Hence, this inequality implies that sequence $\{a_k^{(1)}, a_k^{(2)}, \ldots\}$ is a Cauchy sequence of complex numbers for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is complete, then $\{a_k^{(1)}, a_k^{(2)}, \ldots\}$ converges, say to b_k. Define $\beta = [b_1, b_2, \ldots]^T$. We now show $\beta \in \ell^2(\infty)$ and $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ converges to β.

With the above notation, we have by the Triangle Inequality on $\ell^2(\infty)$ that

$$\left\{\sum_{k=1}^{v} |b_k - a_k^{(n)}|^2\right\}^{1/2} = \left\{\sum_{k=1}^{v} |b_k - a_k^{(m)} + a_k^{(m)} - a_k^{(n)}|^2\right\}^{1/2}$$

$$\leq \left\{\sum_{k=1}^{v} |b_k - a_k^{(m)}|^2\right\}^{1/2} + \left\{\sum_{k=1}^{v} |a_k^{(m)} - a_k^{(n)}|^2\right\}^{1/2} \qquad (4.9)$$

for any $m \in \mathbb{N}$.

Theorem I.4.3 (continued 4)

Proof (continued). Hence, this inequality implies that sequence $\{a_k^{(1)}, a_k^{(2)}, \ldots\}$ is a Cauchy sequence of complex numbers for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is complete, then $\{a_k^{(1)}, a_k^{(2)}, \ldots\}$ converges, say to b_k. Define $\beta = [b_1, b_2, \ldots]^T$. We now show $\beta \in \ell^2(\infty)$ and $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ converges to β.

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for any $m \in \mathbb{N}$.

Theorem I.4.3 (continued 5)

Proof (continued). Since $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ is a Cauchy sequence, for given $\varepsilon > 0$ there is positive $N_0(\varepsilon)$ such that for all $m, n > N_0(\varepsilon)$ and for any $v \in \mathbb{N}$ we have

$$\sum_{k=1}^{v} |a_k^{(m)} - a_k^{(n)}|^2 \leq \|\alpha^{(m)} - \alpha^{(n)}\|^2 < \varepsilon^2/4. \quad (*)$$

Since $b_k = \lim_{m\to\infty} a_k^{(m)}$ for each $k \in \mathbb{N}$, then for any fixed v there is positive $N_v(\varepsilon)$ such that

$$|b_k - a_k^{(m)}| < \varepsilon/2^{(k+1)/2} \text{ for all } m > N_v(\varepsilon) \quad (**)$$

and for all $k = 1, 2, \ldots, v$ (choose such $N(\varepsilon)$ for each of $k = 1, 2, \ldots, v$ and then let $N_v(\varepsilon)$ be the maximum of these $N(\varepsilon)$ for $k = 1, 2, \ldots, v$).

Theorem I.4.3 (continued 5)

Proof (continued). Since $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ is a Cauchy sequence, for given $\varepsilon > 0$ there is positive $N_0(\varepsilon)$ such that for all $m, n > N_0(\varepsilon)$ and for any $v \in \mathbb{N}$ we have

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Theorem I.4.3 (continued 6)

Proof (continued). So from (4.9) we have for all $n > N_0(\varepsilon)$ that

$$\begin{aligned}
\left\{\sum_{k=1}^{v} |b_k - a_k^{(n)}|^2\right\}^{1/2} &\le \left\{\sum_{k=1}^{v} |b_k - a_k^{(m)}|^2\right\}^{1/2} + \left\{\sum_{k=1}^{v} |a_k^{(m)} - a_k^{(n)}|^2\right\}^{1/2} \\
&\le \left\{\sum_{k=1}^{v} \left(\frac{\varepsilon}{2^{(k+1)/2}}\right)^2\right\}^{1/2} + \left(\frac{\varepsilon^2}{4}\right) \text{ by } (*) \text{ and } (**) \\
&= \frac{\varepsilon}{2}\left(\sum_{k=1}^{v} \frac{1}{2^k}\right)^{1/2} + \frac{\varepsilon}{2} \\
&\le \frac{\varepsilon}{2}\left(\sum_{k=1}^{\infty} \frac{1}{2^k}\right)^{1/2} + \frac{\varepsilon}{2} = \varepsilon \quad (4.10)
\end{aligned}$$

Now the right hand side of (4.10) is independent of v, we have that (4.10) holds for all $v \in \mathbb{N}$ where $n > N_0(\varepsilon)$.

Theorem I.4.3 (continued 7)

Proof (continued). So

$$\left\{\sum_{k=1}^{\infty}|b_k - a_k^{(n)}|^2\right\}^{1/2} \leq \varepsilon \text{ for all } n > N_0(\varepsilon). \quad (4.11)$$

Again from the Triangle Inequality in $\ell^2(\infty)$,

$$\begin{aligned}\left\{\sum_{k=1}^{v}|b_k|^2\right\}^{1/2} &= \left\{\sum_{k=1}^{n}|b_k - a_k^{(n)} + a_k^{(n)}|^2\right\}^{1/2} \\ &\leq \left\{\sum_{k=1}^{v}|b_k - a_k^{(n)}|^2\right\}^{1/2} + \left\{\sum_{k=1}^{v}|a_k^{(2)}|^2\right\}^{1/2} \\ &\leq \varepsilon + \left\{\sum_{k=1}^{v}|a_k^{(n)}|^2\right\}^{1/2} \text{ by (4.10).}\end{aligned}$$

Theorem I.4.3 (continued 8)

Proof (continued). Letting $v \to \infty$, this inequality implies $\left\{\sum_{k=1}^{\infty} |b_k|^2\right\}^{1/2} < \infty$ since $\alpha^{(n)} \in \ell^2(\infty)$, and so $\beta \in \ell^2(\infty)$. By (4.11), $\|\beta - \alpha^{(n)}\| \le \varepsilon$ for $n > N_0(\varepsilon)$ and so $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ converges to β. Therefore $\ell^2(\infty)$ is a complete Euclidean space (that is, $\ell^2(\infty)$ is a Hilbert space).

Now for separability. Let D be the set of all elements of $\ell^2(\infty)$ which have a finite number of nonzero components and each nonzero component is a rational complex number (so the nonzero components are of the form $q_1 + q_2 i$ where $q_1, q_2 \in \mathbb{Q}$). Then D is countable (as is to be shown in Exercise I.4.7). Let $\gamma \in \ell^2(\infty)$ where $\gamma = [c_1, c_2, \ldots]^T$.

Theorem I.4.3 (continued 8)

Proof (continued). Letting $v \to \infty$, this inequality implies $\left\{\sum_{k=1}^{\infty} |b_k|^2\right\}^{1/2} < \infty$ since $\alpha^{(n)} \in \ell^2(\infty)$, and so $\beta \in \ell^2(\infty)$. By (4.11), $\|\beta - \alpha^{(n)}\| \leq \varepsilon$ for $n > N_0(\varepsilon)$ and so $\{\alpha^{(1)}, \alpha^{(2)}, \ldots\}$ converges to β. Therefore $\ell^2(\infty)$ is a complete Euclidean space (that is, $\ell^2(\infty)$ is a Hilbert space).

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Theorem I.4.3 (continued 9)

Proof (continued). Then $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ and so for given $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} |c_k|^2 < \varepsilon^2/2$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (and the rational complex numbers are dense in $\mathbb{C}$), then for $k = 1, 2, \ldots, n$ there is rational complex a_k such that $|c_k = a_k| < \varepsilon/\sqrt{2n}$. Let $\alpha = [a_1, a_2, \ldots, a_n, 0, 0, \ldots]^T \in D$. Then

$$\|\gamma - \alpha\| = \left\{ \sum_{k=1}^{n} |c_k = a_k|^2 + \sum_{k=n+1}^{\infty} |c_k|^2 \right\}^{1/2} < \left\{ \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} \right\}^{1/2} = \varepsilon.$$

Therefore countable set D is dense in $\ell^2(\infty)$ and so $\ell^2(\infty)$ is a separable Hilbert space, as claimed. □

Theorem I.4.3 (continued 9)

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Therefore countable set D is dense in $\ell^2(\infty)$ and so $\ell^2(\infty)$ is a separable Hilbert space, as claimed. □

Theorem I.4.5

Theorem I.4.5. A Euclidean space $\mathcal{E}$ is separable if and only if there is a countable orthonormal basis in $\mathcal{E}$.

Proof. First, let $\mathcal{E}$ be a separable Hilbert space. Then (by the definition of separable; Definition I.4.2) there is a countable set $S = \{f_1, f_2, \ldots\}$ which is everywhere dense in $\mathcal{E}$, so that $\overline{S} = \mathcal{E}$. By Theorem I.2.4 there is a countable orthonormal system $T = \{e_1, e_2, \ldots\}$ such that $\mathrm{span}(S) = \mathrm{span}(T)$.

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$$\begin{aligned} [T] &= \overline{(T)} \text{ by Theorem I.4.4} \\ &= \overline{(S)} \text{ since } (S) = \text{span}(S) = \text{span}(T) = (T) \\ &= [S] \text{ by Theorem I.4.4} \\ &= \mathcal{E} \text{ since } \overline{S} = \mathcal{E}. \end{aligned}$$

So T is an orthonormal basis for $\mathcal{E}$, as claimed.

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$$\begin{aligned}
[T] &= \overline{(T)} \text{ by Theorem I.4.4} \\
&= \overline{(S)} \text{ since } (S) = \text{span}(S) = \text{span}(T) = (T) \\
&= [S] \text{ by Theorem I.4.4} \\
&= \mathcal{E} \text{ since } \overline{S} = \mathcal{E}.
\end{aligned}$$

So T is an orthonormal basis for $\mathcal{E}$, as claimed.

Theorem I.4.5 (continued 1)

Proof (continued). Conversely, suppose $T = \{e_1, e_2, \ldots\}$ is a countable orthonormal basis for $\mathcal{E}$. Consider the set

$$R = \{r_1 f_1 + r_2 f_2 + \cdots r_n e_n \mid \mathrm{Re}(r_1), \mathrm{Im}(r_1), \mathrm{Re}(r_2), \mathrm{Im}(r_2),$$

$$\ldots, \mathrm{Re}(r_n), \mathrm{Im}(r_n) \in \mathbb{Q}, \text{ for } n \in \mathbb{N}\}.$$

Then R is countable (Prugovečki mentions Exercise I.4.7 here). Let $\varepsilon > 0$ and $f \in \mathcal{E}$ be given. Since T is an orthonormal basis then by definition (Definition I.4.4) $[T] = \mathcal{E}$ and by Theorem I.4.4, $\overline{\mathrm{span}(T)} = \overline{(T)} = [T] = \mathcal{E}$. So $f \in [T] = \overline{(T)}$ and f is a point of closure of (T). So there is $g \in (T)$ such that $\|f - g\| < \varepsilon/2$. Now g is of the form $g = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$ for some $n \in \mathbb{N}$, so $\|f - a_1 e_1 - a_2 e_2 - \cdots - a_n e_n\| < \varepsilon/2$. Next, for $k = 1, 2, \ldots, n$ there is $r_k \in \mathbb{C}$ where $\mathrm{Re}(r_k), \mathrm{Im}(r_k) \in \mathbb{Q}$ and $|r_k = a_k| < \varepsilon/(2n)$.

Theorem I.4.5 (continued 1)

Proof (continued). Conversely, suppose $T = \{e_1, e_2, \ldots\}$ is a countable orthonormal basis for $\mathcal{E}$. Consider the set

$$R = \{r_1 f_1 + r_2 f_2 + \cdots r_n e_n \mid \mathrm{Re}(r_1), \mathrm{Im}(r_1), \mathrm{Re}(r_2), \mathrm{Im}(r_2),$$

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Theorem I.4.5 (continued 2)

Proof (continued). Let $h = r_1 e_1 + r_2 e_2 + \cdots + r_n e_n \in R$. Then

$$\begin{aligned}
\|f - h\| &= \|f - g + g - h\| \leq \|f - g\| + \|g - h\| \\
&< \varepsilon/2 + \|(a_1 - r_1)e_1 + (a_2 - r_2)e_2 + \cdots + (a_n - r_n)e_n\| \\
&\leq \varepsilon/2 + \sum_{k=1}^{n} |a_k - r_k| \text{ by the Triangle Inequality and} \\
&\quad \text{ the fact that } e_1, e_2, \ldots, e_n \text{ are unit vectors} \\
&= \frac{\varepsilon}{2} + \sum_{k=1}^{n} \frac{\varepsilon}{2n} = \varepsilon.
\end{aligned}$$

So countable set R is dense in $\mathcal{E}$ and $\mathcal{E}$ is separable, as claimed. $\square$

Lemma I.4.1

Lemma I.4.1. For any given vector f in a Euclidean space $\mathcal{E}$ (not necessarily separable) and any countable system $\{e_1, e_2, \ldots\}$ in $\mathcal{E}$, the sequence $\{f_1, f_2, \ldots\}$ of vectors, $f_n = \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k$ is a Cauchy sequence, and the Fourier coefficients $\langle e_k \mid f \rangle$ satisfy Bessel's inequality $\|f_n\| = \sum_{k=1}^{n} |\langle e_k \mid f \rangle|^2 \leq \|f\|^2$.

Proof. Define $h_n = f - f_n$. Then for $i = 1, 2, \ldots, n$

$$\begin{aligned}
\langle e_i \mid h_n \rangle &= \left\langle e_i \mid f - \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k \right\rangle \\
&= \langle e_i \mid r \rangle - \sum_{k=1}^{n} \langle e_k \mid f \rangle \langle e_i \mid e_k \rangle \\
&= \langle e_i \mid f \rangle - \langle e_i \mid f \rangle \text{ since } \langle e_i \mid e_k \rangle = \delta_{ik} \\
&= 0,
\end{aligned}$$

and so ...

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Proof. Define $h_n = f - f_n$. Then for $i = 1, 2, \ldots, n$

$$
\begin{aligned}
\langle e_i \mid h_n \rangle &= \left\langle e_i \mid f - \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k \right\rangle \\
&= \langle e_i \mid r \rangle - \sum_{k=1}^{n} \langle e_k \mid f \rangle \langle e_i \mid e_k \rangle \\
&= \langle e_i \mid f \rangle - \langle e_i \mid f \rangle \text{ since } \langle e_i \mid e_k \rangle = \delta_{ik} \\
&= 0,
\end{aligned}
$$

and so ...

Lemma I.4.1 (continued 1)

Proof (continued). ...

$$\langle f_n \mid h_n \rangle = \left\langle \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k \;\middle|\; h_n \right\rangle = \sum_{k=1}^{n} \langle e_k \mid f \rangle^* \langle e_k \mid h_n \rangle = 0.$$

Thus, $\langle f \mid \rangle = \langle f_n + h_n \mid f_n + h_n \rangle = \langle f_n \mid f_n \rangle + \langle h_n \mid h_n \rangle$ and since $\langle h_n \mid h_n \rangle = \|h_n\|^2 \geq 0$ then $\|f_n\|^2 = \langle f_n \mid f_n \rangle \leq \langle f \mid f \rangle = \|f\|^2$. Also

$$\begin{aligned} \|f_n\|^2 &= \langle f_n \mid f_n \rangle - \left\langle \sum_{i=1}^{n} \langle e_i \mid f \rangle e_i \;\middle|\; \sum_{j=1}^{n} \langle e_i \mid f \rangle e_j \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_i \mid f \rangle^* \langle e_i \mid e_j \rangle \langle e_j \mid f \rangle \\ &= \sum_{i=1}^{n} |\langle e_i \mid f \rangle|^2 \text{ since } \langle e_i \mid e_j \rangle = \delta_{ij}, \ldots \end{aligned}$$

Lemma I.4.1 (continued 1)

Proof (continued). ...

$$\langle f_n \mid h_n\rangle = \left\langle \sum_{k=1}^{n} \langle e_k \mid f\rangle e_k \;\middle|\; h_n \right\rangle = \sum_{k=1}^{n} \langle e_k \mid f\rangle^* \langle e_k \mid h_n\rangle = 0.$$

Thus, $\langle f \mid \rangle = \langle f_n + h_n \mid f_n + h_n\rangle = \langle f_n \mid f_n\rangle + \langle h_n \mid h_n\rangle$ and since $\langle h_n \mid h_n\rangle = \|h_n\|^2 \geq 0$ then $\|f_n\|^2 = \langle f_n \mid f_n\rangle \leq \langle f \mid f\rangle = \|f\|^2$. Also

$$\begin{aligned}
\|f_n\|^2 &= \langle f_n \mid f_n\rangle - \left\langle \sum_{i=1}^{n} \langle e_i \mid f\rangle e_i \;\middle|\; \sum_{j=1}^{n} \langle e_i \mid f\rangle e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_i \mid f\rangle^* \langle e_i \mid e_j\rangle \langle e_j \mid f\rangle \\
&= \sum_{i=1}^{n} |\langle e_i \mid f\rangle|^2 \text{ since } \langle e_i \mid e_j\rangle = \delta_{ij}, \ldots
\end{aligned}$$

Lemma I.4.1 (continued 2)

Proof (continued). ...so $\|f_n\|^2 = \sum_{i=1}^n |\langle e_i \mid f\rangle|^2 \le \|f\|^2$ and Bessel's Inequality holds, as claimed.

Next, since $\|f\|^2$ is finite and $\sum_{i=1}^n |\langle e_i \mid f\rangle|^2 \le \|f\|^2$ for all $n \in \mathbb{N}$ then $\sum_{i=1}^\infty |\langle e_i \mid f\rangle|^2$ converges. So for $\varepsilon > 0$ there is positive $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ we have $\sum_{i=n}^\infty |\langle e_i \mid f\rangle|^2$ (the tail of a convergent series must be "small"). So for $m, n > N(\varepsilon)$ with $m > n$ we have

$$|f_m - f_n\|^2 = \sum_{i=n+1}^{m} |\langle e_i \mid f\rangle|^2 \le \sum_{i=n}^{\infty} |\langle e_i \mid f\rangle|^2 < \varepsilon$$

and so $\{f_1, f_2, \ldots\}$ is a Cauchy sequence, as claimed. $\square$

Lemma I.4.1 (continued 2)

Proof (continued). ...so $\|f_n\|^2 = \sum_{i=1}^n |\langle e_i \mid f\rangle|^2 \le \|f\|^2$ and Bessel's Inequality holds, as claimed.

Next, since $\|f\|^2$ is finite and $\sum_{i=1}^n |\langle e_i \mid f\rangle|^2 \le \|f\|^2$ for all $n \in \mathbb{N}$ then $\sum_{i=1}^\infty |\langle e_i \mid f\rangle|^2$ converges. So for $\varepsilon > 0$ there is positive $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ we have $\sum_{i=n}^\infty |\langle e_i \mid f\rangle|^2$ (the tail of a convergent series must be "small"). So for $m, n > N(\varepsilon)$ with $m > n$ we have

$$|f_m - f_n\|^2 = \sum_{i=n+1}^{m} |\langle e_i \mid f\rangle|^2 \le \sum_{i=n}^{\infty} |\langle e_i \mid f\rangle|^2 < \varepsilon$$

and so $\{f_1, f_2, \ldots\}$ is a Cauchy sequence, as claimed. $\square$

Theorem I.4.6

Theorem I.4.6. Each of the following is a necessary and sufficient condition for a countable orthonormal system $T = \{e_1, e_2, \dots\}$ to be a basis in a separable Hilbert space $\mathcal{H}$.

(a) The only vector f satisfying the relations $\langle e_k \mid f \rangle = 0$ for all $k \in \mathbb{N}$ is the zero vector, $\mathbf{0}$.

(b) For any vector $f \in \mathcal{H}$, $\lim_{n\to\infty} \|f - \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k\| = 0$ or $f = \sum_{k=1}^{\infty} \langle e_k \mid f \rangle e_k$. The $\langle e_k \mid f \rangle$ are *Fourier coefficients* of f with respect to basis T.

(c) Any two vectors $f, g \in \mathcal{H}$ satisfy Parseval's relation $\langle f \mid g \rangle = \sum_{l=1}^{\infty} \langle f \mid e_k \rangle \langle e_k \mid g \rangle$.

(d) For any $f \in \mathcal{H}$, $\|f\| = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2$.

Proof. If T is a countable orthonormal system (not necessarily a basis) in Hilbert space $\mathcal{H}$, then by Lemma I.4.1 for any $f \in \mathcal{H}$ the sequence $\{f_1, f_2, \dots\}$ is Cauchy where $f_n = \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k$. Since $\mathcal{H}$ is complete, this sequence has a limit, say $g \in \mathcal{H}$.

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Proof. If T is a countable orthonormal system (not necessarily a basis) in Hilbert space $\mathcal{H}$, then by Lemma I.4.1 for any $f \in \mathcal{H}$ the sequence $\{f_1, f_2, \ldots\}$ is Cauchy where $f_n = \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k$. Since $\mathcal{H}$ is complete, this sequence has a limit, say $g \in \mathcal{H}$.

Theorem I.4.6 (continued 1)

Proof (continued). $\underline{T \text{ orthonormal basis} \Rightarrow \text{(a)}}$ Let $f \in \mathcal{H}$ be such that $\langle e_k \mid f \rangle = 0$ for all $k \in \mathbb{N}$. By Definition I.4.4 ("orthonormal basis"), $\mathcal{H} = [T] = \overline{(T)}$ and so there is a sequence $\{g_1, g_2, \ldots\} \subset (T)$ which converges to f. Let $g_n = \sum_{k=1}^{s_n} a_k e_k$. Then

$$
\begin{aligned}
\langle f \mid f \rangle &= \left\langle f \;\middle|\; \lim_{n\to\infty} g_n \right\rangle \\
&= \langle f \mid g_n \rangle \text{ by Exercise I.4.10 (with } f_n \text{ and } g_n \text{ of Exercise I.4.10} \\
&\qquad \text{equal to } f \text{ and } g_n \text{ here, respectively)} \\
&= \lim_{n\to\infty} \left\langle f \;\middle|\; \sum_{k=1}^{s_n} a_k e_k \right\rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{s_n} \langle f \mid a_k e_k \rangle \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{s_n} a_k \langle f \mid e_k \rangle \right) \\
&= \lim_{n\to\infty} 0 = 0,
\end{aligned}
$$

so that $f = \mathbf{0}$, as claimed.

Theorem I.4.6 (continued 1)

Proof (continued). T orthonormal basis $\Rightarrow$ (a) Let $f \in \mathcal{H}$ be such that $\langle e_k \mid f \rangle = 0$ for all $k \in \mathbb{N}$. By Definition I.4.4 ("orthonormal basis"), $\mathcal{H} = [T] = \overline{(T)}$ and so there is a sequence $\{g_1, g_2, \ldots\} \subset (T)$ which converges to f. Let $g_n = \sum_{k=1}^{s_n} a_k e_k$. Then

$$\begin{aligned}
\langle f \mid f \rangle &= \left\langle f \,\middle|\, \lim_{n\to\infty} g_n \right\rangle \\
&= \langle f \mid g_n \rangle \text{ by Exercise I.4.10 (with } f_n \text{ and } g_n \text{ of Exercise I.4.10} \\
&\qquad \text{equal to } f \text{ and } g_n \text{ here, respectively)} \\
&= \lim_{n\to\infty} \left\langle f \,\middle|\, \sum_{k=1}^{s_n} a_k e_k \right\rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{s_n} \langle f \mid a_k e_k \rangle \right) \\
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&= \lim_{n\to\infty} 0 = 0,
\end{aligned}$$

so that $f = \mathbf{0}$, as claimed.

Theorem I.4.6 (continued 2)

Proof (continued). (b) $\Rightarrow$ T orthonormal basis Define $f_n = \sum_{k=1}^{n} \langle g_k \mid f \rangle e_k$. Then by (b), $\lim_{n\to\infty} \|f - f_n\| = 0$ and so sequence $\{f_, f_2, \ldots\}$ converges to f. So f is a limit point in $\mathcal{H}$ of (T). That is, $f \in \overline{(T)} = [T]$, so T is an orthonormal basis of $\mathcal{H}$.

(a) $\Rightarrow$ (b) We know sequence $\{f_1, f_2, \ldots\}$, where $f_n = \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k$, converges by the observation above, and

$$\begin{aligned}
\left\langle f - \lim_{n\to\infty} f_n \;\middle|\; e_k \right\rangle &= \left\langle \lim_{n\to\infty} (f - f_n) \;\middle|\; e_k \right\rangle \\
&= \lim_{n\to\infty} \langle f - f_n \mid e_k \rangle \text{ by Exercise I.4.10 (with } f_n \text{ and} \\
&\quad\ g_n \text{ of Exercise I.4.10 replaced with } f - f_n \\
&\quad\ \text{and } e_n \text{ here, respectively)} \\
&= \lim_{n\to\infty} (\langle f \mid e_k \rangle - \langle f_n \mid e_k \rangle) \ldots
\end{aligned}$$

Theorem I.4.6 (continued 2)

Proof (continued). (b) $\Rightarrow$ T orthonormal basis Define $f_n = \sum_{k=1}^{n} \langle g_k \mid f \rangle e_k$. Then by (b), $\lim_{n\to\infty} \|f - f_n\| = 0$ and so sequence $\{f, f_2, \ldots\}$ converges to f. So f is a limit point in $\mathcal{H}$ of (T). That is, $f \in \overline{(T)} = [T]$, so T is an orthonormal basis of $\mathcal{H}$.

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\left\langle f - \lim_{n\to\infty} f_n \,\middle|\, e_k \right\rangle &= \left\langle \lim_{n\to\infty} (f - f_n) \,\middle|\, e_k \right\rangle \\
&= \lim_{n\to\infty} \langle f - f_n \mid e_k \rangle \text{ by Exercise I.4.10 (with } f_n \text{ and} \\
&\quad\ g_n \text{ of Exercise I.4.10 replaced with } f - f_n \\
&\quad\ \text{and } e_n \text{ here, respectively)} \\
&= \lim_{n\to\infty} (\langle f \mid e_k \rangle - \langle f_n \mid e_k \rangle) \ldots
\end{aligned}$$

Theorem I.4.6 (continued 3)

Proof (continued). ...

$$\begin{aligned}
\left\langle f - \lim_{n\to\infty} f_n \;\middle|\; e_k \right\rangle &= \langle f \mid e_k\rangle - \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} \langle e_i \mid f\rangle e_i \;\middle|\; e_k \right\rangle \\
&= \langle f \mid e_k\rangle - \lim_{n\to\infty} \left(\sum_{i=1}^{n} \langle e_i \mid f\rangle^* \langle e_i \mid e_k\rangle \right) \\
&= \langle f \mid e_k\rangle - \langle e_k \mid f\rangle^* \text{ since } \langle e_i \mid e_k\rangle = \delta_{ik} \\
&= \langle f \mid e_k\rangle - \langle f \mid e_k\rangle = 0.
\end{aligned}$$

So by (a), $f - \lim_{n\to\infty} f_n = 0$, or $f - \lim_{n\to\infty} f_n$, as claimed in (b).

So (a) $\Rightarrow$ (b) $\Rightarrow$ T orthonormal basis $\Rightarrow$ (a) and the result holds for (a) and (b).

Theorem I.4.6 (continued 3)

Proof (continued). ...

$$\begin{aligned}
\left\langle f - \lim_{n\to\infty} f_n \;\middle|\; e_k \right\rangle &= \langle f \mid e_k\rangle - \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} \langle e_i \mid f\rangle e_i \;\middle|\; e_k \right\rangle \\
&= \langle f \mid e_k\rangle - \lim_{n\to\infty} \left(\sum_{i=1}^{n} \langle e_i \mid f\rangle^* \langle e_i \mid e_k\rangle \right) \\
&= \langle f \mid e_k\rangle - \langle e_k \mid f\rangle^* \text{ since } \langle e_i \mid e_k\rangle = \delta_{ik} \\
&= \langle f \mid e_k\rangle - \langle f \mid e_k\rangle = 0.
\end{aligned}$$

So by (a), $f - \lim_{n\to\infty} f_n = 0$, or $f - \lim_{n\to\infty} f_n$, as claimed in (b).

So (a) $\Rightarrow$ (b) $\Rightarrow$ T orthonormal basis $\Rightarrow$ (a) and the result holds for (a) and (b).

Theorem I.4.6 (continued 4)

Proof (continued). (b) $\Rightarrow$ (c) By (b), we have for $f, g \in \mathcal{H}$ that $f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \langle e_k \mid f\rangle e_k\right)$ and $g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \langle e_k \mid g\rangle e_k\right)$. So

$$\begin{aligned}
\langle f_n \mid g_n\rangle &= \left\langle \sum_{i=1}^{n} \langle e_i \mid f\rangle e_i \,\middle|\, \sum_{j=1}^{n} \langle e_j \mid g\rangle e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_i \mid f\rangle^* \langle e_j \mid g\rangle \langle e_i \mid e_j\rangle \\
&= \sum_{k=1}^{n} \langle e_k \mid f\rangle^* \langle e_k \mid g\rangle \text{ since } \langle e_i \mid e_j\rangle \delta_{ij} \\
&= \sum_{k=1}^{n} \langle f \mid e_k\rangle \langle e_k \mid g\rangle.
\end{aligned}$$

Theorem I.4.6 (continued 5)

Proof (continued). By Exercise I.4.10,

$$\langle f \mid g \rangle = \lim_{n\to\infty} \langle f_n \mid g_n \rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \langle f \mid e_k \rangle \langle e_k \mid g \rangle \right) = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \langle e_k \mid g \rangle,$$

and so Parseval's relation of (c) holds, as claimed.

(c) $\Rightarrow$ (a) Suppose f is orthogonal to $e_1, e_2, \ldots$. Then by Parseval's relation from (c),

$$\|f\|^2 = \langle f \mid f \rangle = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \langle e_k \mid f \rangle = 0$$

and so $f = \mathbf{0}$ and (a) holds.

Theorem I.4.6 (continued 5)

Proof (continued). By Exercise I.4.10,

$$\langle f \mid g\rangle = \lim_{n\to\infty}\langle f_n \mid g_n\rangle = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\langle f \mid e_k\rangle\langle e_k \mid g\rangle\right) = \sum_{k=1}^{\infty}\langle f \mid e_k\rangle\langle e_k \mid g\rangle,$$

and so Parseval's relation of (c) holds, as claimed.

<u>(c) $\Rightarrow$ (a)</u> Suppose f is orthogonal to $e_1, e_2, \ldots$. Then by Parseval's relation from (c),

$$\|f\|^2 = \langle f \mid f\rangle = \sum_{k=1}^{\infty}\langle f \mid e_k\rangle\langle e_k \mid f\rangle = 0$$

and so $f = \mathbf{0}$ and (a) holds.

Since (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Leftrightarrow$ T orthonormal basis, then the result holds for (a), (b), and (c).

Theorem I.4.6 (continued 5)

Proof (continued). By Exercise I.4.10,

$$\langle f \mid g \rangle = \lim_{n\to\infty} \langle f_n \mid g_n \rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \langle f \mid e_k \rangle \langle e_k \mid g \rangle \right) = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \langle e_k \mid g \rangle,$$

and so Parseval's relation of (c) holds, as claimed.

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Since (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Leftrightarrow$ T orthonormal basis, then the result holds for (a), (b), and (c).

Theorem I.4.6 (continued 6)

Proof (continued). (c) $\Rightarrow$ (d) By Parseval's relation from (c), for $f \in \mathcal{H}$, $\|f\|^2 = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \overline{\langle e_k \mid f \rangle} = \sum_{k=1}^{\infty} \langle e_k \mid f \rangle^* \langle e_k \mid f \rangle = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2$, and (d) holds, as claimed.

(d) $\Rightarrow$ (a) Suppose $\langle e_k \mid f \rangle = 0$ for $k \in \mathbb{N}$. Then by (d), $\|f\|^2 = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2 = 0$ and so $f = \mathbf{0}$ and (a) holds, as claimed.

Theorem I.4.6 (continued 6)

Proof (continued). (c) $\Rightarrow$ (d) By Parseval's relation from (c), for $f \in \mathcal{H}$, $\|f\|^2 = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \overline{\langle e_k \mid f \rangle} = \sum_{k=1}^{\infty} \langle e_k \mid f \rangle^* \langle e_k \mid f \rangle = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2$, and (d) holds, as claimed.

(d) $\Rightarrow$ (a) Suppose $\langle e_k \mid f \rangle = 0$ for $k \in \mathbb{N}$. Then by (d), $\|f\|^2 = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2 = 0$ and so $f = \mathbf{0}$ and (a) holds, as claimed.

Since (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Leftrightarrow$ T orthonormal basis, then the result holds for (a), (b), (c), and (d). □

Theorem I.4.6 (continued 6)

Proof (continued). (c) $\Rightarrow$ (d) By Parseval's relation from (c), for $f \in \mathcal{H}$, $\|f\|^2 = \sum_{k=1}^{\infty} \langle f \mid e_k \rangle \overline{\langle e_k \mid f \rangle} = \sum_{k=1}^{\infty} \langle e_k \mid f \rangle^* \langle e_k \mid f \rangle = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2$, and (d) holds, as claimed.

(d) $\Rightarrow$ (a) Suppose $\langle e_k \mid f \rangle = 0$ for $k \in \mathbb{N}$. Then by (d), $\|f\|^2 = \sum_{k=1}^{\infty} |\langle e_k \mid f \rangle|^2 = 0$ and so $f = \mathbf{0}$ and (a) holds, as claimed.

Since (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Leftrightarrow$ T orthonormal basis, then the result holds for (a), (b), (c), and (d). $\square$

Theorem I.4.7

Theorem I.4.7. Fundamental Theorem of Infinite Dimensional Vector Spaces.
All complex infinite-dimensional separable Hilbert spaces are isomorphic to $\ell^2(\infty)$, and consequently are mutually isomorphic.

Proof. Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space. By Theorem I.4.5, there is an orthonormal countable basis $\{e_1, e_2, \ldots\}$ of $\mathcal{H}$. So by Theorem I.4.6(b) and (d), for any $f \in \mathcal{H}$ we have

$$f = \sum_{k=1}^{\infty} c_k e_k \text{ where } c_k = \langle e_k \mid f \rangle \text{ and } \sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2 < \infty.$$

Therefore $\alpha_f = [e_1, e_2, \ldots]^T \in \ell^2(\infty)$. So we define a mapping $\varphi : \mathcal{H} \to \ell^2(\infty)$ where $\varphi(f) = \alpha_f$.

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Therefore $\alpha_f = [e_1, e_2, \ldots]^T \in \ell^2(\infty)$. So we define a mapping $\varphi : \mathcal{H} \to \ell^2(\infty)$ where $\varphi(f) = \alpha_f$.

Theorem I.4.7 (continued 1)

Proof (continued). Conversely, if $\beta = [b_1, b_2, \ldots]^T \in \ell^2(\infty)$ then the sequence $\{f_1, f_2, \ldots\}$ where $f_n = \sum_{k=1}^n b_k e_k$ is a Cauchy sequence since for any $\varepsilon > 0$ there is positive $N(\varepsilon)$ such that for $n > N(\varepsilon)$ we have $\sum_{k=n}^\infty |b_k|^2 < \varepsilon$ (because $\beta \in \ell^2(\infty)$), and so for $m, n > N(\varepsilon)$ where $m > n$ we have

$$\|f_m - f_n\| = \sum_{k=n+1}^{m} |b_k|^2 \leq \sum_{k=n}^{\infty} |b_k|^2 < \varepsilon.$$

Since $\mathcal{H}$ is complete, then Cauchy sequence $\{f_1, f_2, \ldots\}$ converges to some (unique) $f \in \mathcal{H}$. Also,

$$\begin{aligned} \langle e_k \mid f \rangle &= \left\langle e_k \;\middle|\; \lim_{n\to\infty}\left(\sum_{i=1}^{n} b_i e_i\right)\right\rangle \\ &= \lim_{n\to\infty}\left\langle e_k \;\middle|\; \sum_{i=1}^{n} b_i e_i\right\rangle \text{ by Exercise I.4.10} \ldots \end{aligned}$$

Theorem I.4.7 (continued 1)

Proof (continued). Conversely, if $\beta = [b_1, b_2, \ldots]^T \in \ell^2(\infty)$ then the sequence $\{f_1, f_2, \ldots\}$ where $f_n = \sum_{k=1}^n b_k e_k$ is a Cauchy sequence since for any $\varepsilon > 0$ there is positive $N(\varepsilon)$ such that for $n > N(\varepsilon)$ we have $\sum_{k=n}^\infty |b_k|^2 < \varepsilon$ (because $\beta \in \ell^2(\infty)$), and so for $m, n > N(\varepsilon)$ where $m > n$ we have

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Since $\mathcal{H}$ is complete, then Cauchy sequence $\{f_1, f_2, \ldots\}$ converges to some (unique) $f \in \mathcal{H}$. Also,

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Theorem I.4.7 (continued 1)

Proof (continued). ...

$$\langle e_k \mid f \rangle \;=\; \lim_{n\to\infty}\left(\sum_{i=1}^{n} b_i \langle e_k \mid e_i \rangle\right) = b_k.$$

So the mapping $\varphi : \mathcal{H} \to \ell^2(\infty)$ defined above has an inverse and φ is one to one and onto. It is to be shown that mapping φ is an inner product space isomorphism (that is, the three parts of Definition I.2.4 are satisfied). □

Theorem I.4.8

Theorem I.4.8. Let $\mathcal{E}$ be a separable Euclidean space with an orthonormal basis $\{e_1, e_2, \ldots\}$ and let $\mathcal{E}'$ be a Euclidean space. If there is a unitary transformation from $\mathcal{E}$ to $\mathcal{E}'$ (that is, $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic inner product spaces) and if e_n transforms to e_n', then $\{e_1', e_2', \ldots\}$ is an orthonormal basis in $\mathcal{E}'$.

Proof. Let $\mathcal{E}$ be infinite dimensional and denote by $\langle \cdot \mid \cdot \rangle_1$ and $\langle \cdot \mid \cdot \rangle_2$ the inner products on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Since the unitary transformation (i.e., isomorphism) preserves inner products, then $\langle e_i' \mid e_j' \rangle_2 = \langle e_i \mid e_j \rangle_1 = \delta_{ij}$ and so $\{e_1', e_2', \ldots\}$ is an orthonormal system in $\mathcal{E}'$. For each $f' \in \mathcal{E}'$, there is a unique $f \in \mathcal{E}$ such that the unitary transformation maps $f \mapsto f'$.

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$$\lim_{n\to\infty} \left\| f' - \sum_{k=1}^{n} \langle e_k' \mid f' \rangle_2 e_k' \right\|_2 = \lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \langle e_k \mid f \rangle_1 e_k \right\|_1 = 0.$$

Theorem I.4.8

Theorem I.4.8. Let $\mathcal{E}$ be a separable Euclidean space with an orthonormal basis $\{e_1, e_2, \ldots\}$ and let $\mathcal{E}'$ be a Euclidean space. If there is a unitary transformation from $\mathcal{E}$ to $\mathcal{E}'$ (that is, $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic inner product spaces) and if e_n transforms to e_n', then $\{e_1', e_2', \ldots\}$ is an orthonormal basis in $\mathcal{E}'$.

Proof. Let $\mathcal{E}$ be infinite dimensional and denote by $\langle \cdot \mid \cdot \rangle_1$ and $\langle \cdot \mid \cdot \rangle_2$ the inner products on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Since the unitary transformation (i.e., isomorphism) preserves inner products, then $\langle e_i' \mid e_j' \rangle_2 = \langle e_i \mid e_j \rangle_1 = \delta_{ij}$ and so $\{e_1', e_2', \ldots\}$ is an orthonormal system in $\mathcal{E}'$. For each $f' \in \mathcal{E}'$, there is a unique $f \in \mathcal{E}$ such that the unitary transformation maps $f \mapsto f'$. Now the unitary transformation also preserves norms so

$$\lim_{n\to\infty} \left\| f' - \sum_{k=1}^{n} \langle e_k' \mid f' \rangle_2 e_k' \right\|_2 = \lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \langle e_k \mid f \rangle_1 e_k \right\|_1 = 0.$$

Theorem I.4.8

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Proof. So by Theorem I.4.6(b), $\{e'_1, e'_2, \ldots\}$ is a basis of $\mathcal{E}'$, as claimed.

If $\mathcal{E}$ is finite dimensional, the proof is similar (just drop the limits). $\square$

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Proof. So by Theorem I.4.6(b), $\{e_1', e_2', \ldots\}$ is a basis of $\mathcal{E}'$, as claimed.

If $\mathcal{E}$ is finite dimensional, the proof is similar (just drop the limits). $\square$