Chapter I. Basic Ideas of Hilbert Space Theory
I.5. Wave Mechanics of a Single Particle Moving in One Dimension—Proofs of Theorems
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Theorem I.5.A. Schroedinger’s equation implies that \( \| \psi(x, t) \| \) is a constant with respect to time \( t \) where for each fixed \( t \),
\[
\lim_{x \to \pm \infty} \psi(x, t) = 0 \quad \text{and} \quad \lim_{x \to \pm \infty} \left( \frac{\partial \psi(x, t)}{\partial x} \right) = 0.
\]

Proof. We justify the claim by showing \( \frac{d}{dt} [\| \psi(x, t) \|^2] = 0 \). We have
\[
\frac{d}{dt} [\| \psi(x, t) \|^2] = \frac{d}{dt} \left[ \int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx \right] = \frac{d}{dt} \left[ \int_{-\infty}^{\infty} \psi^*(x, t) \psi(x, t) \, dx \right]
\]
\[
= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} [\psi^*(x, t) \psi(x, t)] \, dx \quad \text{by Leibniz’s Rule}
\]
\[
= \int_{-\infty}^{\infty} \left( \frac{\partial \psi^*(x, t)}{\partial t} \psi(x, t) + \psi^*(x, t) \frac{\partial \psi(x, t)}{\partial t} \right) \, dx
\]
\[
= \int_{-\infty}^{\infty} \left\{ \left( \frac{\hbar}{2mi} \frac{\partial \psi^*(x, t)}{\partial x^2} - \frac{1}{i\hbar} V(x) \psi^*(x, t) \right) \psi(x, t) 
+ \psi^*(x, t) \left( - \frac{\hbar}{2mi} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{1}{i\hbar} V(x) \psi(x, t) \right) \right\} \, dx \ldots
\]
Theorem I.5.A

Theorem I.5.A. Schroedinger’s equation implies that \( \| \psi(x, t) \| \) is a constant with respect to time \( t \) where for each fixed \( t \),

\[
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\[
\frac{d}{dt} [\| \psi(x, t) \|^2] = \frac{d}{dt} \left[ \int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx \right] = \frac{d}{dt} \left[ \int_{-\infty}^{\infty} \psi^*(x, t) \psi(x, t) \, dx \right]
\]

\[
= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} [\psi^*(x, t) \psi(x, t)] \, dx \quad \text{by Leibniz’s Rule}
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{\partial \psi^*(x, t)}{\partial t} \psi(x, t) + \psi^*(x, t) \frac{\partial \psi(x, t)}{\partial t} \right) \, dx
\]

\[
= \int_{-\infty}^{\infty} \left\{ \left( \frac{\hbar}{2mi} \frac{\partial \psi^*(x, t)}{\partial x^2} - \frac{1}{i\hbar} V(x) \psi^*(x, t) \right) \psi(x, t) \right. \\
+ \psi^*(x, t) \left( - \frac{\hbar}{2mi} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{1}{i\hbar} V(x) \psi(x, t) \right) \right\} \, dx \ldots
\]
Theorem I.5.A (continued 1)

Proof (continued).

by the conjugate of Schroedinger’s equation for \( \frac{\partial \psi^*(x, t)}{\partial t} \)

and Schroedinger’s equation for \( \frac{\partial \psi(x, t)}{\partial t} \)

\[
\begin{align*}
\frac{\hbar}{2mi} \int_{-\infty}^{\infty} & \left( \frac{\partial^2 \psi^*(x, t)}{\partial x^2} \psi(x, t) = \psi^*(x, t) \frac{\partial^2 \psi(x, t)}{\partial x^2} \right) \, dx \\
= & \frac{\hbar}{2mi} \int_{-\infty}^{\infty} \left( \frac{\partial^2 \psi^*}{\partial x^2} \psi + \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial^2 \psi}{\partial x^2} \right) \, dx \\
= & \frac{\hbar}{2mi} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[ \frac{\partial \psi^*}{\partial x} \psi - \psi^* \frac{\partial \psi}{\partial x} \right] \, dx \\
= & \left. \frac{\hbar}{2mi} \left( \frac{\partial \psi^*}{\partial x} \psi - \psi^* \frac{\partial \psi}{\partial x} \right) \right|_{-\infty}^{\infty} \\
= & 0 \text{ since } \lim_{x \to \pm \infty} \psi(x, t) = \lim_{x \to \pm \infty} \frac{\partial \psi(x, t)}{\partial x} = 0.
\end{align*}
\]
Theorem 1.5.A. Schroedinger’s equation implies that $\|\psi(x, t)\|$ is a constant with respect to time $t$ where for each fixed $t$, 
\[ \lim_{x \to \pm \infty} \psi(x, t) = 0 \quad \text{and} \quad \lim_{x \to \pm \infty} \left( \frac{\partial \psi(x, t)}{\partial x} \right) = 0. \]

Proof (continued). So \( \frac{d}{dt}[\|\psi(x, t)\|^2] = 0 \) and $\|\psi(x, t)\|^2$ is a constant real valued continuous function of $t$. That is, $\|\psi(x, t)\|$ is constant with respect to $t$, as claimed.
Theorem I.5.1

Theorem I.5.1. If $\psi_1(x)$ and $\psi_2(x)$, their first derivatives $d\psi_1(x)/dx$ and $d\psi_2(x)/dx$, as well as $V(x)\psi_1(x)$ and $V(x)\psi_2(x)$ are from $C^1_{(2)}(\mathbb{R})$, then

$$\left\langle \psi_1(x) \left| -\frac{\hbar^2}{2m} \frac{d^2\psi_2(x)}{dx^2} + V(x)\psi_2(x) \right. \right\rangle$$

$$= \left\langle -\frac{\hbar^2}{2m} \frac{d^2\psi_1(x)}{dx^2} + V(x)\psi_1(x) \left| \psi_2(x) \right. \right\rangle.$$ 

In each solution $\psi(x)$ of the time-independent Schroedinger equation (5.7) has the property that $\psi(x), d\psi(x)/dx, V(x)\psi(x) \in C^1_{(2)}(\mathbb{R})$, then each eigenvalue $E$ of the time-independent Schroedinger equation is a real number, and if $\psi_1(x)$ and $\psi_2(x)$ are two eigenfunctions of the time-independent Schroedinger equation corresponding to two distinct eigenvalues $E_1 \neq E_2$, then $\psi_1(x)$ and $\psi_2(x)$ are orthogonal.
Theorem I.5.1 (continued 1)

**Proof.** We have (not writing the variable $x$):

\[
\int_{-a}^{a} \psi_1^* \left( -\frac{\hbar^2}{2m} \frac{d^2 \psi_2}{dx^2} + V \psi_2 \right) \, dx
\]

\[
= \int_{-a}^{a} -\frac{\hbar^2}{2m} \psi_1^* \frac{d^2 \psi_2}{dx^2} \, dx + \int_{-a}^{a} V \psi_1^* \psi_2 \, dx
\]

let $u = -\frac{\hbar^2}{2m} \psi_1^*$ and $dv = \frac{d^2 \psi_2}{dx^2} \, dx$

so $du = -\frac{\hbar^2}{2m} \frac{d \psi_1^*}{dx} \, dx$ and $v = \frac{d \psi_2}{dx}$

\[
= -\frac{\hbar^2}{2m} \psi_1^* \frac{d \psi_2}{dx} \bigg|_{-a}^{a} - \int_{-a}^{a} -\frac{\hbar^2}{2m} \psi_1^* \frac{d \psi_2}{dx} \, dx + \int_{-a}^{a} V \psi_1^* \psi_2 \, dx \ldots
\]
Theorem I.5.1 (continued 2)

Proof (continued). ...

\[
= - \frac{\hbar^2}{2m} \psi^* \frac{d\psi_2}{dx} \bigg|_a - \int_{-a}^a - \frac{\hbar^2}{2m} \frac{d\psi_1^*}{dx} \frac{d\psi_2}{dx} \ dx + \int_{-a}^a V \psi_1^* \psi_2 \ dx
\]

let \( u = \frac{d\psi_1^*}{dx} \) and \( dv = \frac{d\psi_2}{dx} \)

so \( du = \frac{d^2\psi_1^*}{dx^2} \) and \( v = \psi_2 \)

\[
= \left( - \frac{\hbar^2}{2m} \psi_1^* \frac{d\psi_2}{dx} + \frac{\hbar^2}{2m} \frac{d\psi_1^*}{dx} \psi_2 \right) \bigg|_{-a}^a
\]

\[
= - \frac{\hbar^2}{2m} \psi_2 \frac{d^2\psi_1^*}{dx^2} \bigg|_{-a}^a + \int_{-a}^a V \psi_1^* \psi_2 \ dx
\]

\[
= - \frac{\hbar^2}{2m} \left( \psi_1^* \frac{d\psi_2}{dx} - \frac{d\psi_1^*}{dx} \psi_2 \right) \bigg|_{-a}^a + \int_{-a}^a \left( - \frac{\hbar^2}{2m} \frac{d^2\psi_1^*}{dx^2} + V \psi_1^* \right) \psi_2 \ dx.
\]
Theorem I.5.1 (continued 3)

**Proof (continued).** Since $\psi_1, \psi_2 \in C^1_2(\mathbb{R})$ then $\lim_{x \to \pm \infty} \psi_1(x) = \lim_{x \to \pm \infty} \psi_2(x) = \lim_{x \to \pm \infty} \frac{d\psi_1(x)}{dx} = \lim_{x \to \pm \infty} \frac{d\psi_2(x)}{dx} = 0$ and so

$$\lim_{a \to \infty} \left[ -\frac{\hbar^2}{2m} \left( \psi_1^* \frac{d\psi_2}{dx} - \frac{d\psi_1^*}{dx} \psi_2 \right) \right]_{-a}^a = 0.$$  

If we know $d^2\psi_1/dx^2, d^2\psi_2/dx^2 \in C^1_2(\mathbb{R})$ then we know the inner product is defined and so the limit as $a \to \infty$ of the two integrals above exist, so that

$$\lim_{a \to \infty} \left( \int_{-a}^a \psi_1^* \left( -\frac{\hbar^2}{2m} \frac{d^2\psi_2}{dx^2} + V \psi_2 \right) dx \right)$$

$$= \lim_{a \to \infty} \left( \int_{-a}^a \left( -\frac{\hbar}{2m} d^2\psi_1^* dx^2 + V \psi_1^* \right) dx \right)$$

so the inner product claim of the theorem holds.
Theorem I.5.1 (continued 3)

Proof (continued). Since $\psi_1, \psi_2 \in C^1(\mathbb{R})$ then $\lim_{x \to \pm \infty} \psi_1(x) = \lim_{x \to \pm \infty} \psi_2(x) = \lim_{x \to \pm \infty} \frac{d\psi_1(x)}{dx} = \lim_{x \to \pm \infty} \frac{d\psi_2(x)}{dx} = 0$ and so

$$\lim_{a \to \infty} \left. \left( \psi_1^* \psi_2 - \frac{d\psi_1^*}{dx} \psi_2 \right) \right|_{-a}^{a} = 0.$$ 

If we know $d^2\psi_1/dx^2, d^2\psi_2/dx^2 \in C^1(\mathbb{R})$ then we know the inner product is defined and so the limit as $a \to \infty$ of the two integrals above exist, so that

$$\lim_{a \to \infty} \left( \int_{-a}^{a} \psi_1^* \left( -\frac{\hbar^2}{2m} \frac{d^2\psi_2}{dx^2} + V\psi_2 \right) \, dx \right)$$

$$= \lim_{a \to \infty} \left( \int_{-a}^{a} \left( -\frac{\hbar}{2m} d^2\psi_1^* dx^2 + V\psi_1^* \right) \, dx \right)$$

so the inner product claim of the theorem holds. (BUT it does seem that we need the added assumption $d^2\psi_1/dx^2, d^2\psi_2/dx^2 \in C^1(\mathbb{R})$ in order to insure convergence of the above integrals.)

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Theorem I.5.1 (continued 3)

Proof (continued). Since \( \psi_1, \psi_2 \in C^1_2(\mathbb{R}) \) then \( \lim_{x \to \pm \infty} \psi_1(x) = \lim_{x \to \pm \infty} \psi_2(x) = \lim_{x \to \pm \infty} \frac{d\psi_1(x)}{dx} = \lim_{x \to \pm \infty} \frac{d\psi_2(x)}{dx} = 0 \) and so

\[
\lim_{a \to \infty} \frac{\hbar^2}{2m} \left( \psi_1^* \frac{d\psi_2}{dx} - \frac{d\psi_1^*}{dx} \psi_2 \right) \bigg|_{-a}^{a} = 0.
\]

If we know \( d^2\psi_1/dx^2, d^2\psi_2/dx^2 \in C^1_2(\mathbb{R}) \) then we know the inner product is defined and so the limit as \( a \to \infty \) of the two integrals above exist, so that

\[
\lim_{a \to \infty} \left( \int_{-a}^{a} \psi_1^* \left( -\frac{\hbar^2}{2m} \frac{d^2\psi_2}{dx^2} + V \psi_2 \right) dx \right)
\]

\[
\lim_{a \to \infty} \left( \int_{-a}^{a} \left( -\frac{\hbar}{2m} d^2\psi_1^* dx^2 + V \psi_1^* \right) dx \right)
\]

so the inner product claim of the theorem holds. (BUT it does seem that we need the added assumption \( d^2\psi_1/dx^2, d^2\psi_2/dx^2 \in C^1_2(\mathbb{R}) \) in order to insure convergence of the above integrals.)
Proof (continued). If we know that $\psi_1$ and $\psi_2$ satisfy the time-independent Schrödinger equation then we know

$$-\frac{\hbar^2}{2m} \frac{d^2\psi_i}{dx^2} = E_i \psi_i - V \psi_i \in C^1_{(2)}(\mathbb{R}) \text{ for } i = 1, 2.$$ 

If $\psi_1(x)$ and $\psi_2(x)$ satisfy the conditions of the theorem and if in addition they are solutions of the time-independent Schrödinger equation with eigenvalues $E_1$ and $E_2$ (so that the issue raised about $d^2\psi_1/dx^2$ and $d^2\psi_2/dx^2$ are not a concern) then by the first part of the theorem concerning inner products and from the time-independent Schrödinger equation we conclude $\langle \psi_1(x) | E_2 \psi_2(x) \rangle = \langle E_1 \psi_1(x) | \psi_2(x) \rangle$. 
Proof (continued). If we know that $\psi_1$ and $\psi_2$ satisfy the time-independent Schrödinger equation then we know
\[
\frac{-\hbar^2}{2m} \frac{d^2\psi_i}{dx^2} = E_i\psi_i - V\psi_i \in C_1(\mathbb{R}) \text{ for } i = 1, 2.
\]
If $\psi_1(x)$ and $\psi_2(x)$ satisfy the conditions of the theorem and if in addition they are solutions of the time-independent Schrödinger equation with eigenvalues $E_1$ and $E_2$ (so that the issue raised about $d^2\psi_1/dx^2$ and $d^2\psi_2/dx^2$ are not a concern) then by the first part of the theorem concerning inner products and from the time-independent Schrödinger equation we conclude $\langle \psi_1(x) | E_2\psi_2(x) \rangle = \langle E_1\psi_1(x) | \psi_2(x) \rangle$. If we take $\psi_1(x) = \psi_2(x) = \psi(x)$ and $E_1 = E_2 = E$ in this equation, then we get
\[
E\langle \psi | \psi \rangle = \langle \psi | E\psi \rangle = \langle E\psi | \psi \rangle = E^*\langle \psi | \psi \rangle.
\]
So if $\psi$ is a nontrivial solution of the time-independent Schrödinger equation (i.e., $\psi(x) \neq 0$) then $\|\psi\|^2 = \langle \psi | \psi \rangle > 0$ and so $E = E^*$ in this case. That is, the eigenvalues of the time-independent Schrödinger equation are real.
Proof (continued). If we know that $\psi_1$ and $\psi_2$ satisfy the
time-independent Schrödinger equation then we know
\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi_i}{dx^2} = E_i \psi_i - V \psi_i \in C^1_2(\mathbb{R}) \text{ for } i = 1, 2.
\]
If $\psi_1(x)$ and $\psi_2(x)$ satisfy the conditions of the theorem and if in addition
they are solutions of the time-independent Schrödinger equation with
eigenvalues $E_1$ and $E_2$ (so that the issue raised about $d^2 \psi_1/dx^2$ and
$d^2 \psi_2/dx^2$ are not a concern) then by the first part of the theorem
concerning inner products and from the time-independent Schrödinger
equation we conclude
\[
\langle \psi_1(x) | E_2 \psi_2(x) \rangle = \langle E_1 \psi_1(x) | \psi_2(x) \rangle.
\]
If we take $\psi_1(x) = \psi_2(x) = \psi(x)$ and $E_1 = E_2 = E$ in this equation, then we get
\[
E \langle \psi | \psi \rangle = \langle \psi | E \psi \rangle = \langle E \psi | \psi \rangle = E^* \langle \psi | \psi \rangle.
\]
So if $\psi$ is a nontrivial solution of the time-independent Schrödinger equation (i.e., $\psi(x) \neq 0$)
then $\|\psi\|^2 = \langle \psi | \psi \rangle > 0$ and so $E = E^*$ in this case. That is, the
eigenvalues of the time-independent Schrödinger equation are real.
Proof (continued). Finally, let $\psi_1$ and $\psi_2$ be solutions to the time-independent Schrödinger equation with associated eigenvalues $E_1$ and $E_2$, respectively, where $E_1 \neq E_2$. Then $\langle \psi_1 | E_2 \psi_2 \rangle = \langle E_1 \psi_1 | \psi_2 \rangle$, or (since $E_1$ and $E_2$ are real) $E_2 \langle \psi_1 | \psi_2 \rangle = E_1 \langle \psi_1 | \psi_2 \rangle$, or $(E_2 - E_1)\langle \psi_1 | \psi_2 \rangle = 0$. Since $E_1 \neq E_2$ then we must have $\psi_1 \perp \psi_2$, as claimed. \qed
Theorem I.5.B

Theorem I.5.B. Let $\mathcal{H}_b^{(1)}$ be the (topologically) closed subspace of $\mathcal{H}^{(1)}$ which is spanned by $\mathcal{E}_b$ (where $\mathcal{E}_b$ is the set of “bound states”; that is, the set of $C_{(2)}^1(\mathbb{R})$ which are solutions of the time-independent Schroedinger equations). Then an orthonormal basis of $\mathcal{H}_b^{(1)}$ is given by $T = \bigcup_{E \in S_p} T_E$ where $T_E$ is an orthonormal basis for $M_E$. NOTE: You may assume that $\mathcal{H}^{(1)}$ is separable (as will be shown in Chapter II).

Proof. First, since we assume $\mathcal{H}^{(1)}$ is separable, then by Theorem I.4.2 $\mathcal{H}_b^{(1)}$ is also separable. For each $E \in S_b$, $M_E$ is a subspace of $\mathcal{H}_b^{(1)}$ and so $M_E$ is also separable. By Theorem I.4.5, each $M_E$ has an at most countably infinite orthonormal system $T_E$ spanning $M_E$. 
Theorem I.5.B

**Theorem I.5.B.** Let $\mathcal{H}^{(1)}_b$ be the (topologically) closed subspace of $\mathcal{H}^{(1)}$ which is spanned by $\mathcal{E}_b$ (where $\mathcal{E}_b$ is the set of “bound states”; that is, the set of $C^1_{(2)}(\mathbb{R})$ which are solutions of the time-independent Schrödinger equations). Then an orthonormal basis of $\mathcal{H}^{(1)}_b$ is given by $T = \bigcup_{E \in S_p} T_E$ where $T_E$ is an orthonormal basis for $M_E$. NOTE: You may assume that $\mathcal{H}^{(1)}$ is separable (as will be shown in Chapter II).

**Proof.** First, since we assume $\mathcal{H}^{(1)}$ is separable, then by Theorem I.4.2 $\mathcal{H}^{(1)}_b$ is also separable. For each $E \in S_b$, $M_E$ is a subspace of $\mathcal{H}^{(1)}_b$ and so $M_E$ is also separable. By Theorem I.4.5, each $M_E$ has an at most countably infinite orthonormal system $T_E$ spanning $M_E$. Consider $T = \bigcup_{E \in S_p} T_E$. Now every element of $\mathcal{E}_b$ is some (countable) sum of elements of $T$. Since the closure of the span of $\mathcal{E}_b$ is $\mathcal{H}^{(1)}_b$, then the closed subspace spanned by $T$ is $\mathcal{H}^{(1)}_b$. 
Theorem I.5.B

**Theorem I.5.B.** Let $\mathcal{H}_b^{(1)}$ be the (topologically) closed subspace of $\mathcal{H}^{(1)}$ which is spanned by $\mathcal{E}_b$ (where $\mathcal{E}_b$ is the set of “bound states”; that is, the set of $C^1_2(\mathbb{R})$ which are solutions of the time-independent Schroedinger equations). Then an orthonormal basis of $\mathcal{H}_b^{(1)}$ is given by $T = \cup_{E \in S} T_E$ where $T_E$ is an orthonormal basis for $M_E$. NOTE: You may assume that $\mathcal{H}^{(1)}$ is separable (as will be shown in Chapter II).

**Proof.** First, since we assume $\mathcal{H}^{(1)}$ is separable, then by Theorem I.4.2 $\mathcal{H}_b^{(1)}$ is also separable. For each $E \in S_b$, $M_E$ is a subspace of $\mathcal{H}_b^{(1)}$ and so $M_E$ is also separable. By Theorem I.4.5, each $M_E$ has an at most countably infinite orthonormal system $T_E$ spanning $M_E$. Consider $T = \cup_{E \in S} T_E$. Now every element of $\mathcal{E}_b$ is some (countable) sum of elements of $T$. Since the closure of the span of $\mathcal{E}_b$ is $\mathcal{H}_b^{(1)}$, then the closed subspace spanned by $T$ is $\mathcal{H}_b^{(1)}$. 
Theorem 1.5.B (continued)

Theorem 1.5.B. Let $\mathcal{H}_b^{(1)}$ be the (topologically) closed subspace of $\mathcal{H}^{(1)}$ which is spanned by $\mathcal{E}_b$ (where $\mathcal{E}_b$ is the set of “bound states”; that is, the set of $C^1_{(2)}(\mathbb{R})$ which are solutions of the time-independent Schroedinger equations). Then an orthonormal basis of $\mathcal{H}_b^{(1)}$ is given by $T = \bigcup_{E \in S_p} T_E$ where $T_E$ is an orthonormal basis for $M_E$. NOTE: You may assume that $\mathcal{H}^{(1)}$ is separable (as will be shown in Chapter II).

Proof (continued). Since each $T_E$ is orthonormal and by Theorem I.5.1 every element of $T_{E_1}$ is orthonormal to every subset of $T_{E_2}$ for $E_1 \neq E_2$, then set $T$ is orthonormal. So $T$ is linearly independent (see Exercise I.4.12) and by Exercise I.5.3, $T$ is countable. So $T$ is an orthonormal basis of $\mathcal{H}_b^{(1)}$, as claimed. □
Theorem I.5.2

Theorem I.5.2. For any fixed $t \in \mathbb{R}$, the sequence $\{\Phi_1(t), \Phi_2(t), \ldots\}$,

$$\Phi_n(t) = \sum_{k=1}^{n} c_k(t) \psi_k$$

where $c_k(t) = \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \langle \psi_k | \psi_0 \rangle$,

is convergent in the norm of $\mathcal{H}^{(1)}_b$ to some $\psi(t) \in \mathcal{H}^{(1)}_b$. For $t = t_0$, $\lim_{n \to \infty} \Phi_n(t_0) = \psi(t_0)$ satisfies the initial condition $\psi(t_0) = \psi_0$.

Proof. Since $\{\psi_1, \psi_2, \ldots\}$ is an orthonormal basis in $\mathcal{H}^{(1)}_b$, by Theorem I.4.6(d),

$$\sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k=1}^{\infty} \left| \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \langle \psi_k | \psi_0 \rangle \right|^2$$
Theorem 1.5.2

**Theorem 1.5.2.** For any fixed $t \in \mathbb{R}$, the sequence $\{\Phi_1(t), \Phi_2(t), \ldots\}$, 

$$\Phi_n(t) = \sum_{k=1}^{n} c_k(t) \psi_k$$

where $c_k(t) = \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \langle \psi_k | \psi_0 \rangle$, 

is convergent in the norm of $\mathcal{H}^{(1)}_b$ to some $\psi(t) \in \mathcal{H}^{(1)}_b$. For $t = t_0$, $\lim_{n \to \infty} \Phi_n(t_0) = \psi(t_0)$ satisfies the initial condition $\psi(t_0) = \psi_0$.

**Proof.** Since $\{\psi_1, \psi_2, \ldots\}$ is an orthonormal basis in $\mathcal{H}^{(1)}_b$, by Theorem I.4.6(d),

$$\sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k=1}^{\infty} \left| \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \langle \psi_k | \psi_0 \rangle \right|^2$$
Theorem I.5.2 (continued 1)

Proof (continued).

\[
\sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k=1}^{\infty} |\langle \psi_k | \psi_0 \rangle|^2 \quad \text{since} \quad \left| \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \right| = 1
\]

because \( E_k \) is real by Theorem I.5.1

\[
= \| \psi_0 \|^2 < \infty.
\]

So for given \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( \sum_{k=N}^{\infty} |c_k(t)|^2 < \varepsilon^2 \). So if \( m, n > N \) (with \( m \geq n \), say) then

\[
|\Phi_m(t) - \Phi_n(t)|^2 = \left| \sum_{k=n}^{m} c_k(t) \psi_k \right|^2 = \sum_{k=n}^{m} |c_k(t)|^2 \leq \sum_{k=N}^{\infty} |c_k(t)|^2 < \varepsilon^2,
\]

and so \( \{ \Phi_n(t) \} \) is a Cauchy sequence in \( \mathcal{H}_b^{(1)} \) and hence, since \( \mathcal{H}_b^{(1)} \) is complete, converges to some \( \Psi(t) \in \mathcal{H}_b^{(1)} \).
Theorem I.5.2 (continued 1)

Proof (continued).

\[
\sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k=1}^{\infty} |\langle \psi_k | \psi_0 \rangle|^2 \text{ since } \left| \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \right| = 1
\]

because \( E_k \) is real by Theorem I.5.1

\[
= \| \psi_0 \|^2 < \infty.
\]

So for given \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( \sum_{k=N}^{\infty} |c_k(t)|^2 < \varepsilon^2 \). So if \( m, n > N \) (with \( m \geq n \), say) then

\[
|\Phi_m(t) - \Phi_n(t)|^2 = \left| \sum_{k=n}^{m} c_k(t) \psi_k \right|^2 = \sum_{k=n}^{m} |c_k(t)|^2 \leq \sum_{k=N}^{\infty} |c_k(t)|^2 < \varepsilon^2,
\]

and so \( \{ \Phi_n(t) \} \) is a Cauchy sequence in \( \mathcal{H}_b^{(1)} \) and hence, since \( \mathcal{H}_b^{(1)} \) is complete, converges to some \( \Psi(t) \in \mathcal{H}_b^{(1)} \).
Theorem I.5.2. For any fixed $t \in \mathbb{R}$, the sequence $\{\Phi_1(t), \Phi_2(t), \ldots\}$,

$$\Phi_n(t) = \sum_{k=1}^{n} c_k(t) \psi_k$$

where $c_k(t) = \exp \left( -\frac{i}{\hbar} E_k(t - t_0) \right) \langle \psi_k | \psi_0 \rangle$,

is convergent in the norm of $\mathcal{H}_b^{(1)}$ to some $\psi(t) \in \mathcal{H}_b^{(1)}$. For $t = t_0$, $\lim_{n \to \infty} \Phi_n(t_0) = \psi(t_0)$ satisfies the initial condition $\psi(t_0) = \psi_0$.

Proof (continued). With $t = t_0$, $c_k = \langle \psi_k | \psi_0 \rangle$ for $k \in \mathbb{N}$ and so

$$\psi(t_0) = \lim_{n \to \infty} \Phi_n(t_0) = \lim_{n \to \infty} \sum_{k=1}^{n} c_k(t_0) \psi_k = \sum_{k=1}^{\infty} c_k(t_0) \psi_k = \sum_{k=1}^{\infty} \langle \psi_k | \psi_0 \rangle \psi_k$$

by Theorem I.4.6, as claimed.
Theorem I.5.C. Suppose the series

$$
\sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \Psi_k \mid \Psi_0 \rangle \varphi_k(x)
$$

converges in the $H^{(1)}_b$ norm for each fixed value of $t$ and converges pointwise for each value of $x$ and $t$ to a limit function $\varphi(x, t)$, and that $\frac{\partial^2 \varphi(x, t)}{\partial x^2}$ and $\frac{\partial \psi(x, t)}{\partial t}$ can be obtained by differentiating the series term by term twice in $x$ and once in $t$. Here, $\varphi_k(x)$ satisfies the time-independent Schrödinger equation for $E = E_k$. Then $\varphi(x, t)$ is a solution to Schrödinger’s equation

$$
i\hbar \frac{\partial \varphi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi(x, t)}{\partial x^2} + V(x) \varphi(x, t).$$
Proof. By the hypotheses on differentiability, we have

\[
\frac{\partial \varphi(x, t)}{\partial t} = -\frac{i}{\hbar} \sum_{k=1}^{\infty} E_k \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \frac{d \varphi_k(x)}{dt}
\]

\[
\frac{\partial^2 \varphi(x, t)}{\partial x^2} = \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \frac{d^2 \varphi_k(x)}{dx^2}
\]

(notice the inner product is an integral over \(x\) so \(\langle \psi_k | \psi_0 \rangle\) is a constant).

So

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi(x, t)}{\partial x^2} + V(x) \varphi(x, t) = \frac{\hbar}{2m} \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \frac{d^2 \varphi_k(x)}{dx^2}
\]

\[
V(x) \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \varphi_k(x)
\]
Theorem I.5.C (continued)

Proof. By the hypotheses on differentiability, we have

\[
\frac{\partial \varphi(x, t)}{\partial t} = -\frac{i}{\hbar} \sum_{k=1}^{\infty} E_k \exp \left(-\frac{i}{\hbar} E_k (t - t_0)\right) \langle \psi_k | \psi_0 \rangle \frac{d\varphi_k(x)}{dt}
\]

\[
\frac{\partial^2 \varphi(x, t)}{\partial x^2} = \sum_{k=1}^{\infty} \exp \left(-\frac{i}{\hbar} E_k (t - t_0)\right) \langle \psi_k | \psi_0 \rangle \frac{d^2 \varphi_k(x)}{dx^2}
\]

(notice the inner product is an integral over \(x\) so \(\langle \psi_k | \psi_0 \rangle\) is a constant). So

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi(x, t)}{\partial x^2} + V(x) \varphi(x, t)
\]

\[
= -\frac{\hbar}{2m} \sum_{k=1}^{\infty} \exp \left(-\frac{i}{\hbar} E_k (t - t_0)\right) \langle \psi_k | \psi_0 \rangle \frac{d^2 \varphi_k(x)}{dx^2}
\]

\[
V(x) \sum_{k=1}^{\infty} \exp \left(-\frac{i}{\hbar} E_k (t - t_0)\right) \langle \psi_k | \psi_0 \rangle \varphi_k(x)
\]
Theorem I.5.C (continued)

Proof (continued).

\[
\sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \left( -\frac{\hbar^2}{2m} \frac{d^2 \varphi_k(x)}{dx^2} + V(x) \varphi_k(x) \right)
\]

\[
= \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle E_k \varphi_k(x)
\]

since \( \varphi_k(x) \) is a solution to the time-independent Schroedinger equation for \( E = E_k \):

\[
-\frac{\hbar^2}{2m} \frac{d^2 \varphi_k(x)}{dx^2} + V(x) \varphi_k(x) = E_k \varphi_k(x)
\]

So

\[
\varphi(x, t) = \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k | \psi_0 \rangle \varphi_k(x)
\]

is a solution to the Schroedinger equation, as claimed. \(\square\)
Theorem I.5.C (continued)

Proof (continued).

\[
= \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k \mid \psi_0 \rangle \left( -\frac{\hbar^2}{2m} \frac{d^2 \varphi_k(x)}{dx^2} + V(x) \varphi_k(x) \right)
\]

\[
= \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k \mid \psi_0 \rangle E_k \varphi_k(x)
\]

since \( \varphi_k(x) \) is a solution to the time-independent Schroedinger equation for \( E = E_k : -\frac{\hbar^2}{2m} \frac{d^2 \varphi_k(x)}{dx^2} + V(x) \varphi_k(x) = E_k \varphi_k(x) \)

\[
= \frac{\hbar}{-i} \frac{\partial \varphi(x, t)}{\partial t} = i\frac{\hbar}{-i} \frac{\partial \varphi(x, t)}{\partial t} = i\hbar \frac{\partial \varphi(x, t)}{\partial t}.
\]

So

\[
\varphi(x, t) = \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} E_k (t - t_0) \right) \langle \psi_k \mid \psi_0 \rangle \varphi_k(x)
\]

is a solution to the Schroedinger equation, as claimed.
Theorem 1.5.D. The general solution of
\[ \frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi(x) = 0 \] for \(0 \leq x \leq L\)

\[ \frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} (E - V_0)\psi(x) = 0 \] for \(x < 0, x > L\)

is

\[ \psi(x) = \begin{cases} 
ce^{ikx} + de^{-ikx} & \text{where } k = \sqrt{2mE/\hbar} \text{ for } 0 \leq x \leq L \\
 a_1 e^{-ik'x} + b_1 e^{ik'x} & \text{where } k' = \sqrt{2m(E - V_0)/\hbar} \text{ for } x < 0 \\
 a_2 e^{ik''x} + b_2 e^{-ik''x} & \text{where } k'' = \sqrt{2m(E - V_0)/\hbar} \text{ for } x > L.
\]  

Proof. Since the ODE is second order linear and in each of the three regions \(\psi(x)\) is a linear combination of two linearly independent functions, we just need to confirm that \(\psi(x)\) satisfies the ODE in each region.
Theorem I.5.D. The general solution of

\[ \frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi(x) = 0 \text{ for } 0 \leq x \leq L \]

\[ \frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} (E - V_0)\psi(x) = 0 \text{ for } x < 0, x > L \]

is

\[ \psi(x) = \begin{cases} 
  ce^{ikx} + de^{-ikx} \text{ where } k = \sqrt{\frac{2mE}{\hbar}} \text{ for } 0 \leq x \leq L \\
  a_1 e^{-ik'x} + b_1 e^{ik'x} \text{ where } k' = \sqrt{\frac{2m(E - V_0)}{\hbar}} \text{ for } x < 0 \\
  a_2 e^{ik''x} + b_2 e^{-ik''x} \text{ where } k'' = \sqrt{\frac{2m(E - V_0)}{\hbar}} \text{ for } x > L.
\end{cases} \]

Proof. Since the ODE is second order linear and in each of the three regions \( \psi(x) \) is a linear combination of two linearly independent functions, we just need to confirm that \( \psi(x) \) satisfies the ODE in each region.
Proof (continued). For $0 \leq x \leq L$ we have

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} E \psi(x) = -k^2 ce^{ikx} - k^2 de^{-kx} + \frac{2m}{\hbar^2} E (ce^{ikx} + de^{-ikx})$$

$$= - \frac{2mE}{\hbar^2} ce^{ikx} - \frac{2mE}{\hbar^2} de^{-ikx} + \frac{2m}{\hbar^2} Ece^{ikx} + \frac{2m}{\hbar^2} de^{-ikx} = 0.$$  

For $x < 0$ (and similarly for $x > L$),

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} E \psi(x) = -(k')^2 a_1 e^{-ik'x} - (k')^2 b_1 e^{-k'x}$$

$$+ \frac{2m}{\hbar^2} (E - V_0)(a_1 e^{-ik'x} + b_1 e^{ik'x})$$

$$= - \frac{2m(E - V_0)}{\hbar^2} a_1 e^{-ik'x} - \frac{2m(E - V_0)}{\hbar^2} b_1 e^{-k'x}$$

$$+ \frac{2m(E - V_0)}{\hbar^2} a_1 e^{-ik'x} + \frac{2m(E - V_0)}{\hbar^2} b_1 e^{ik'x} = 0.$$
Theorem I.5.D (continued)

Proof (continued). For $0 \leq x \leq L$ we have

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi(x) = -k^2 ce^{ikx} - k^2 de^{-kx} + \frac{2m}{\hbar^2} E(ce^{ikx} + de^{-ikx})$$

$$= -\frac{2mE}{\hbar^2} ce^{ikx} - \frac{2mE}{\hbar^2} de^{-ikx} + \frac{2m}{\hbar^2} Ece^{ikx} + \frac{2m}{\hbar^2} de^{-ikx} = 0.$$

For $x < 0$ (and similarly for $x > L$),

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi(x) = -(k')^2 a_1 e^{-ik'x} - (k')^2 b_1 e^{-k'x}$$

$$+ \frac{2m}{\hbar^2} (E - V_0)(a_1 e^{-ik'x} + b_1 e^{ik'x})$$

$$= -\frac{2m(E - V_0)}{\hbar^2} a_1 e^{-ik'x} - \frac{2m(E - V_0)}{\hbar^2} b_1 e^{-k'x}$$

$$+ \frac{2m(E - V_0)}{\hbar^2} a_1 e^{-ik'x} + \frac{2m(E - V_0)}{\hbar^2} b_1 e^{ik'x} = 0.$$