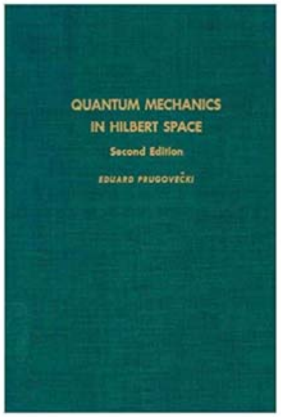
Modern Algebra

Chapter II. Measure Theory and Hilbert Spaces of Functions

II.1. Measurable Spaces—Proofs of Theorems



Lemma II.1.1

Lemma II.1.1. If $\mathscr{F}$ is a family of sets and R is any given set, then

$$R \cap (\cup_{S \in \mathscr{F}} S) = \cup_{S \in \mathscr{F}} (R \cap S).$$

Proof. If $\xi \in R \cap (\cup_{S \in \mathscr{F}} S)$ then $\xi \in R$ and $\xi \in \cup_{S \in \mathscr{F}} S$. That is, $\xi \in T$ for some $T \in \mathscr{F}$. Then $\xi \in R \cap T$ where $T \in \mathscr{F}$ and so $\xi \in \cup_{S \in \mathscr{F}} (R \cap S)$.

Conversely, if $\eta \in \cup_{S \in \mathscr{F}} (R \cap S)$, then $\eta \in R \cap T$ for some $T \in \mathscr{F}$. The $\eta \in R$ and $\eta \in T$ where $T \in \mathscr{F}$. Then $\eta \in R$ and $\eta \in T$, so that $\eta \in \cup_{S \in \mathscr{F}} S$. Therefore $\eta \in R \cap (\cup_{S \in \mathscr{F}} S)$.

Hence $R \cap (\cup_{S \in \mathscr{F}} S) = \cup_{S \in \mathscr{F}} (R \cap S)$, as claimed. $\square$

Lemma II.1.2. DeMorgan's Laws

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If $\mathscr{F}$ is a family of subsets of a set $\mathscr{X}$, and if for any given set S we denote by $S' = \mathscr{X} \setminus S$ the *complement* of S with respect to $\mathscr{X}$, then

$$(\cup_{S \in \mathscr{F}} S)' = \cap_{S \in \mathscr{F}} S' \text{ and } (\cap_{S \in \mathscr{F}} S)' = \cup_{S \in \mathscr{F}} S'.$$

Proof. To establish the first claim, let $\xi \in (\cup_{S \in \mathscr{F}} S)'$. Then $\xi \notin \cup_{S \in \mathscr{F}} S$ and so $\xi \notin S$ for all $S \in \mathscr{F}$. That is, $\xi \in S'$ for all $S \in \mathscr{F}$ and so $\xi \in \cap_{S \in \mathscr{F}} S'$. Hence $(\cup_{S \in \mathscr{F}} S)' \subset \cap_{S \in \mathscr{F}} S'$.

Conversely, if $\eta \in \cap_{S \in \mathscr{F}} S'$ then $\eta \in S'$ for all $S \in \mathscr{F}$. That is, $\eta \notin S$ for all $S \in \mathscr{F}$. Therefore $\eta \notin \cup_{S \in \mathscr{F}} S$ and so $\eta \in (\cup_{S \in \mathscr{F}} S)'$. Hence $\cap_{S \in \mathscr{F}} S' \subset (\cup_{S \in \mathscr{F}} S)'$ and so $(\cup_{S \in \mathscr{F}} S)' = \cap_{S \in \mathscr{F}} S'$, as claimed.

Lemma II.1.2 (continued)

Lemma II.1.2. DeMorgan's Laws.
If $\mathscr{F}$ is a family of subsets of a set $\mathscr{X}$, and if for any given set S we denote by $S' = \mathscr{X} \setminus S$ the *complement* of S with respect to $\mathscr{X}$, then

$$(\cup_{S \in \mathscr{F}} S)' = \cap_{S \in \mathscr{F}} S' \text{ and } (\cap_{S \in \mathscr{F}} S)' = \cup_{S \in \mathscr{F}} S'.$$

Proof (continued). We can take a short cut to prove the second claim. Define $\mathscr{F}' = \{S' \mid S \in \mathscr{F}\}$. Then applying the first claim to family $\mathscr{F}'$ we have $(\cup_{S' \in \mathscr{F}'} S')' = \cap_{S' \in \mathscr{F}'} S'$ and taking complements of both sides

$$\left((\cup_{S' \in \mathscr{F}'} S')' \right)' = (\cap_{S' \in \mathscr{F}'} S')' \text{ or } \cup_{S' \in \mathscr{F}'} S' = (\cap_{S' \in \mathscr{F}'} S')'$$

(since $(R')' = R'' = R$ for any set R). Replacing S' with S (and $\mathscr{F}$ with $\mathscr{F}$) gives $\cup_{S \in \mathscr{F}} S = (\cap_{S \in \mathscr{F}} S)'$, as claimed. $\square$

Theorem II.1.1

Theorem II.1.1. If the class $\mathscr{K}$ of subsets of a set $\mathscr{X}$ is a Boolean algebra, then

(a) the entire set $\mathscr{X}$ and the empty set $\varnothing$ belong to $\mathscr{K}$,

(b) the intersection $R \cap S$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$, and

(c) the difference $R \setminus S$ and symmetric difference $R \triangle S = (R \setminus S) \cup (S \setminus R)$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$.

Proof. 1. If $R \subset \mathscr{X}$ and $R \in \mathscr{K}$, then $R' \in \mathscr{K}$ and so $R \cup R' = \mathscr{X} \in \mathscr{K}$, as claimed. Then $\mathscr{X}' = \varnothing \in \mathscr{K}$, as claimed.

2. For $R, S \in \mathscr{K}$ we have $R \cup S \in \mathscr{K}$ and $(R \cup S)' = R' \cap S' \in \mathscr{K}$ (by Lemma II.1.1), as claimed.

Theorem II.1.1 (continued)

Theorem II.1.1. If the class $\mathscr{K}$ of subsets of a set $\mathscr{X}$ is a Boolean algebra, then

(a) the entire set $\mathscr{X}$ and the empty set $\varnothing$ belong to $\mathscr{K}$,

(b) the intersection $R \cap S$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$, and

(c) the difference $R \setminus S$ and symmetric difference $R \triangle S = (R \setminus S) \cup (S \setminus R)$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$.

Proof (continued). 3. For $R, S \in \mathscr{K}$ we have

$$\begin{aligned} R \setminus S &= R \cap S' \\ &= (R' \cup S'')' \text{ by Lemma II.1.2} \\ &= (R' \cup S)' \\ &\in \mathscr{K} \text{ since } \mathscr{K} \text{ is a Boolean algebra.} \end{aligned}$$

Similarly, $S \setminus R \in \mathscr{K}$. Hence $R \triangle S = (R \setminus S) \cup (S \setminus R) \in \mathscr{K}$, as claimed. □

Theorem II.1.2

Theorem II.1.2. For any given nonempty family $\mathscr{F}$ of subset of a set $\mathscr{X}$ there is a unique smallest Boolean algebra $\mathscr{A}(\mathscr{F})$ and a unique smallest Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{F})$ containing $\mathscr{F}$. That is, if $\mathscr{A}$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$ and if $\mathscr{A}_\sigma$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}_\sigma(\mathscr{F}) \subset \mathscr{A}_\sigma$. $\mathscr{A}(\mathscr{F})$ and $\mathscr{A}_\sigma(\mathscr{F})$ are called, respectively, the *Boolean algebra* and the *Boolean σ algebra generated* by the family $\mathscr{F}$.

Proof. Denote by $\mathfrak{F}$ the family of all Boolean algebras $\mathscr{A}$ containing $\mathscr{F}$. $\mathfrak{F}$ is not empty because it contains the power set $\mathfrak{S}_{\mathscr{X}}$ of all subsets of $\mathscr{X}$. Consider the family $\mathscr{A}(\mathscr{F}) = \cap_{\mathfrak{S} \in \mathfrak{F}} \mathfrak{S}$. Now $\mathscr{A}(\mathscr{F})$ is nonempty since $\mathscr{F} \subset \mathscr{A}(\mathscr{F})$. If $R, S \in \mathscr{A}(\mathscr{F})$ then $R, S \in \mathfrak{S}$ for all $\mathfrak{S} \in \mathfrak{F}$ and since each $\mathfrak{G}$ is an algebra then $R \cup S \in \mathfrak{S}$ and $R' = \mathscr{X} \setminus R \in \mathfrak{S}$ for all $\mathfrak{S} \in \mathfrak{G}$. Therefore $R \cup S \in \mathscr{A}(\mathscr{F})$, $R' \in \mathscr{A}(\mathscr{F})$, and $\mathscr{A}(\mathscr{F} \in \mathfrak{F}$. If $\mathscr{A}$ is any Boolean algebra containing $\mathscr{F}$ then $\mathscr{A} \in \mathfrak{F}$ and so $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$; that is, $\mathscr{A}(\mathscr{F})$ is a smallest Boolean algebra containing $\mathscr{F}$, as claimed.

Theorem II.1.2 (continued)

Theorem II.1.2. For any given nonempty family $\mathscr{F}$ of subset of a set $\mathscr{X}$ there is a unique smallest Boolean algebra $\mathscr{A}(\mathscr{F})$ and a unique smallest Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{F})$ containing $\mathscr{F}$. That is, if $\mathscr{A}$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$ and if $\mathscr{A}_\sigma$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}_\sigma(\mathscr{F}) \subset \mathscr{A}_\sigma$. $\mathscr{A}(\mathscr{F})$ and $\mathscr{A}_\sigma(\mathscr{F})$ are called, respectively, the *Boolean algebra* and the *Boolean σ algebra generated* by the family $\mathscr{F}$.

Proof (continued). For uniqueness, if $\mathscr{A}_1(\mathscr{F})$ and $\mathscr{A}_2(\mathscr{F})$ are two such algebras, then $\mathscr{A}_1(\mathscr{F}) \subset \mathscr{A}_2(\mathscr{F})$ since $\mathscr{A}_1(\mathscr{F})$ is a smallest algebra and $\mathscr{A}_2(\mathscr{F}) \subset \mathscr{A}_1(\mathscr{F})$ since $\mathscr{A}_2(\mathscr{F})$ is a smallest algebra. So $\mathscr{A}_1(\mathscr{F}) = \mathscr{A}_2(\mathscr{F})$ and the smallest such algebra is unique.

The proof for a smallest Boolean σ algebra is similar. □

Theorem II.1.3

Theorem II.1.3. The family $\mathscr{B}_0^n$ of all finite unions

$$I_1 \cup I_2 \cup \cdots \cup I_k \text{ where } I_1, I_2, \ldots, I_n \in \mathscr{I}^n \text{ and } k \in \mathbb{N}$$

of intervals in $\mathscr{I}^n$ is identical to the Boolean algebra $\mathscr{A}(\mathscr{I}^n)$.

Proof. Since a Boolean algebra is closed under finite unions, then $\mathscr{B}_0^n \subset \mathscr{A}(\mathscr{I}^n)$. Now if we show that $\mathscr{B}_0^n$ is a Boolean algebra then we must have $\mathscr{A}(\mathscr{I}^n) \subset \mathscr{B}_0^n$ (since $\mathscr{A}(\mathscr{I}^n)$ is the smallest Boolean algebra containing $\mathscr{I}^n$) and hence $\mathscr{B}_0^n = \mathscr{A}(\mathscr{I}^n)$. If $R, S \in \mathscr{B}_0^n$ then $R = I_1 \cup I_2 \cup \cdots \cup I_k$ and $S = J_1 \cup J_2 \cup \cdots \cup J_\ell$ for some $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_\ell \in \mathscr{B}_0^n$.

To prove $R' \in \mathscr{B}_0^n$, we proceed by induction on k where $R = I_1 \cup I_2 \cup \cdots \cup I_k$. In the case $k = 1$ we have $R = I_1$ is an interval. In Exercise II.1.A it is to be shown that $I_1' = I^{(1)} \cup I^{(2)} \cup \cdots \cup I^{(v)}$ where $I^{(1)}, I^{(2)}, \ldots, I^{(v)} \in \mathscr{I}^n$ and $v \le 2^n 3^{n-1}$. So $R' \in \mathscr{B}_0^n$ and $\mathscr{B}_0^n$ is closed under complements of intervals.

Theorem II.1.3 (continued 1)

Proof (continued). Suppose $R' \in \mathscr{B}_0^n$ for all $R = I_1 \cup I_2 \cup \cdots \cup I_k$ for intervals $I_1, I_2, \ldots, I_k \in \mathscr{I}^n$. Then

$$R' = (I_1 \cup I_2 \cup \cdots \cup I_k)' = \cup_{m=1}^p J_m \qquad (1.5)$$

for some $p \in \mathbb{N}$ and $J_1, J_2, \ldots, J_p \in \mathscr{I}^n$ (since the result holds for $k = 1$ instead), and we have

$$\begin{aligned}
&(I_1 \cup I_2 \cup \cdots \cup I_{k+1})' \\
= \ & (I_1 \cup I_2 \cup \cdots \cup I_k)' \cap I_{k+1} \text{ by Lemma II.1.2} \\
= \ & (\cup_{m=1}^p J_m) \cap (J^{(1)} \cup J^{(2)} \cup \cdots \cup J^{(v)}) \text{ by (1.5)} \\
= \ & \left((\cup_{m=1}^p J_m) \cap J^{(1)}\right) \cup \left((\cup_{m=1}^p J_m) \cap J^{(2)}\right) \cup \cdots \cup \left((\cup_{m=1}^p J_m) \cap J^{(v)}\right) \\
& \text{by Lemma II.1.1} \\
= \ & \left(\cup_{m=1}^p J_m \cap J^{(1)}\right) \cup \left(\cup_{m=1}^p J_m \cap J^{(2)}\right) \cup \cdots \cup \left(\cup_{m=1}^p J_m \cap J^{(v)}\right) \\
& \text{by Lemma II.1.1.}
\end{aligned}$$

Theorem II.1.3 (continued 2)

Theorem II.1.3. The family $\mathscr{B}_0^n$ of all finite unions

$$I_1 \cup I_2 \cup \cdots \cup I_k \text{ where } I_1, I_2, \ldots, I_n \in \mathscr{I}^n \text{ and } k \in \mathbb{N}$$

of intervals in $\mathscr{I}^n$ is identical to the Boolean algebra $\mathscr{A}(\mathscr{I}^n)$.

Proof (continued). Now an intersection of two elements of $\mathscr{I}^n$ is an element of $\mathscr{I}^n$, so $J_m \cap J^{(1)}, J_m \cap J^{(2)}, \ldots, J_m \cap J^{(v)} \in \mathscr{I}^n$ and so $R' = (I_1 \cup I_2 \cup \cdots \cup I_{k+1}) \in \mathscr{B}_0^n$ and so by Mathematical Induction $\mathscr{B}_0^n$ is closed under complements. Therefore $\mathscr{B}_0^n$ is a Boolean algebra and the claim holds, as explained above. □

Theorem II.1.4

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof. Assume O is an open set in $\mathbb{R}^n$. For each $m \in \mathbb{N}$ consider the open intervals in $\mathbb{R}^n$

$$\begin{aligned}
I_{k_1,k_2,\ldots,k_n}^{(m)} &= \left\{ x = (x_1, x_2, \ldots, x_n) \,\middle|\, \frac{k_1 - 1}{m} < x_1 < \frac{k_1 + 1}{m}, \right. \\
&\qquad \left. \frac{k_2 - 1}{m} < x_2 < \frac{k_2 + 1}{m}, \ldots, \frac{k_n - 1}{m} < x_n < \frac{k_n + 1}{m} \right\} \\
&= \left[\frac{k_1 - 1}{m}, \frac{k_1 + 1}{m}\right] \times \left[\frac{k_2 - 1}{m}, \frac{k_2 + 1}{m}\right] \times \cdots \times \left[\frac{k_n - 1}{m}, \frac{k_n + 1}{m}\right]
\end{aligned}$$

for $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Then this countable collection of intervals covers $\mathbb{R}^n$. Consider the collection of all such intervals lying within O:

$$R^{(m)} = \{I_{k_1,k_2,\ldots,k_n}^{(m)} \mid I_{k_1,k_2,\ldots,k_n}^{(m)} \subset O \text{ where } k_1, k_2, \ldots, k_n \in \mathbb{N}\}.$$

Then $R^{(m)}$ is countable for each $m \in \mathbb{Z}$.

Theorem II.1.4 (continued)

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof (continued). Now if $x \in O$ then there is an ε-neighborhood of x contained in O (using the Euclidean metric on $\mathbb{R}^n$ to define such a neighborhood) and so for m sufficiently large (namely, $m > 2n/\varepsilon$) there is an interval $I^{(m)}_{k_1,k_2,\ldots,k_n}$ containing x and lying in the ε neighborhood. Now let A be the union of the intervals in the $\mathbb{R}^{(m)}$, $A = \cup_{m\in\mathbb{Z}, I\in R^{(m)}} I$. Since each element of each $R^{(m)}$ is a subset of O, then $A \subset O$. Since each $x \in O$ is in some element of some $R^{(m)}$ then $O \subset A$. So $O = A$ and O is a countable union of intervals. Since the Borel sets are in the σ algebra generated by $\mathscr{I}^n$, then O is a Borel set. Since any closed set C has an open complement and a σ algebra is closed under complements, then each closed set in $\mathbb{R}^n$ is also Borel. $\square$

Theorem II.1.6

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof. (a) If $\mathscr{A}_\sigma$ is a Boolean σ algebra and $R_1, R_2, \ldots \in \mathscr{A}_\sigma$ is monotonically increasing then $\lim_{k\to\infty} R_k = \sup_{k=1}^\infty R_k \in \mathscr{A}_\sigma$. In the case that $S_1, S_2, \ldots$ is a monotonically decreasing sequence in $\mathscr{A}_\sigma$ and so $\lim_{k\to\infty}(S_1 \setminus S_k) = \cup_{k=1}^\infty (S_1 \setminus S_k) \in \mathscr{A}_\sigma$, where by Lemmas II.1.1 and II.1.2,

$$\cup_{k=1}^\infty (S_1 \setminus S_k) = \cup_{k=1}^\infty (S_1 \cap S_k') = S_1 \cap (\cup_{k=1}^\infty S_k')\ S_1 \setminus (\cup_{k=1}^\infty S_k)'$$

$$= S_1 \setminus (\cap_{k=1}^\infty S_k) = S_1 \setminus \lim_{k\to\infty} S_k.$$

So $S_1 \setminus \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$ and since $\mathscr{A}_\sigma$ is closed under set differences (and S_1 contains $S_2, S_3, \ldots$) then $S_1 \setminus (S_1 \setminus \lim_{k\to\infty} S_k) = \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$. So $\mathscr{A}_\sigma$ is a monotone class, as claimed.

Theorem II.1.6 (continued)

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof (continued). (b) If $\mathscr{A}$ is a Boolean algebra and a monotone class and $S_1, S_2, \ldots$ is an infinite sequence of sets from $\mathscr{A}$, then with $R_n = \sup_{k=1}^n S_k$, the sequence $R_1, R_2, \ldots$ is a monotonically increasing sequence in $\mathbb{A}$. Since $\mathscr{A}$ is a monotone class, then $\cup_{k=1}^\infty S_k = \cup_{k=1}^\infty R_k = \lim_{n\to\infty} R_n \in \mathscr{A}$. So $\mathscr{A}$ is a Boolean algebra closed under countable unions and hence $\mathscr{A}$ is a Boolean σ algebra, as claimed. $\square$

Theorem II.1.7

Theorem II.1.7. If $\mathscr{A}$ is a Boolean algebra and $\mathfrak{M}(\mathscr{A})$ is the monotone class generated by $\mathscr{A}$, then $\mathfrak{M}(\mathscr{A})$ is identical with the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by the family $\mathscr{A}$ of sets.

Proof. By Theorem II.1.6, $\mathscr{A}_\sigma(\mathscr{A})$ is a monotone class and by definition $\mathfrak{M}(\mathscr{A})$ is the smallest monotone class containing $\mathscr{A}$, so $\mathfrak{M}(\mathscr{A}) \subset \mathscr{A}_\sigma(\mathscr{A})$. We will show that $\mathfrak{M}(\mathscr{A})$ is a Boolean σ algebra containing $\mathscr{A}$. Since $\mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, then this will imply $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$ and hence $\mathscr{A}_\sigma(\mathscr{A}) = \mathfrak{M}(\mathscr{A})$.

For $R \in \mathfrak{M}(\mathscr{A})$, denote by $\mathfrak{N}(R)$ the family of sets $S \in \mathfrak{M}(\mathscr{A})$ such that $S', R \cup S \in \mathfrak{M}(\mathscr{A})$. So by definition, $\mathfrak{N}(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$. If $S_1, S_2, \ldots \in \mathfrak{N}(R)$ is a monotone sequence then, since $\mathfrak{M}(\mathscr{A})$ is a monotone class, by Exercise II.1.6 $(\lim_{n\to\infty} S_n)'' = \lim_{n\to\infty} S_n' \in \mathfrak{M}(\mathscr{A})$, and the sequence of sets $R \cup S_1, R \cup S_2, \ldots \in \mathfrak{M}(\mathscr{A})$ is also a monotone sequence and, again by Exercise II.1.6, $(\lim_{n\to\infty} S_n) \cup R = \lim_{n\to\infty}(S_n \cup R) \in \mathfrak{M}(\mathscr{A})$. So, by the definition of $\mathfrak{N}(R)$, $\lim_{n\to\infty} S_n \in \mathfrak{N}(R)$ and so $\mathfrak{N}(R)$ is a monotone class.

Theorem II.1.7 (continued 1)

Proof (continued). In addition, if $R \in \mathscr{A}$ then $\mathfrak{N}(R)$ is not empty because it contains $\mathscr{A}$, since $\mathscr{A}$ is a Boolean algebra. In this case, $\mathfrak{N}(R)$ is a monotone class containing $\mathscr{A}$ ad hence $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(R)$. Therefore $\mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$ for $R \in \mathscr{A}$. Since $\mathscr{A}$ is a Boolean algebra of sets then $\mathscr{X} = \mathfrak{M}(\mathscr{A})$. By the definition of $\mathfrak{N}(\mathscr{X})$, for every $S \in \mathfrak{N}(\mathscr{X}) = \mathfrak{M}(\mathscr{A})$ we have $S' \in \mathfrak{M}(\mathscr{A})$ so that $\mathfrak{M}(\mathscr{A})$ is closed under complements.

Furthermore, if $R \in \mathscr{A}$, then for an arbitrary $S \in \mathfrak{M}(\mathscr{A})$ we have $S \in \mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$, i.e., $R \cup S \in \mathfrak{M}(\mathscr{A})$; consequently (by the definition of $\mathfrak{N}(S)$), $R \in \mathfrak{N}(S)$. Therefore $\mathscr{A} \subset \mathfrak{N}(S)$. As shown above, $\mathfrak{N}(S)$ is a monotone class and, since it contains $\mathscr{A}$, we have $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(S)$. Since $\mathfrak{N}(S) \subset \mathfrak{M}(\mathscr{A})$ by definition, then $\mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ where S is any set in $\mathfrak{M}(\mathscr{A})$. So for any $R, S \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ we have (by the definition of $\mathfrak{N}(S)$) that $R \cup S, R' \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$. Therefore, $\mathfrak{M}(\mathscr{A})$ is a Boolean algebra.

Theorem II.1.7 (continued 2)

Theorem II.1.7. If $\mathscr{A}$ is a Boolean algebra and $\mathfrak{M}(\mathscr{A})$ is the monotone class generated by $\mathscr{A}$, then $\mathfrak{M}(\mathscr{A})$ is identical with the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by the family $\mathscr{A}$ of sets.

Proof (continued). Since $\mathfrak{M}(\mathscr{A})$ is a Boolean algebra which is a monotone class then, by Theorem II.1.6, $\mathfrak{M}(\mathscr{A})$ is a Boolean σ algebra. The result now follows, as explained above. □