Modern Algebra

Chapter II. Measure Theory and Hilbert Spaces of Functions
II.1. Measurable Spaces—Proofs of Theorems

Lemma II.1.1

Lemma II.1.1. If $\mathcal{F}$ is a family of sets and $R$ is any given set, then

$$R \cap (\bigcup_{S \in \mathcal{F}} S) = \bigcup_{S \in \mathcal{F}} (R \cap S).$$

Proof. If $\xi \in R \cap (\bigcup_{S \in \mathcal{F}} S)$ then $\xi \in R$ and $\xi \in \bigcup_{S \in \mathcal{F}} S$. That is, $\xi \in T$ for some $T \in \mathcal{F}$. Then $\xi \in R \cap T$ where $T \in \mathcal{F}$ and so $\xi \in \bigcup_{S \in \mathcal{F}} (R \cap S)$.

Conversely, if $\eta \in \bigcup_{S \in \mathcal{F}} (R \cap S)$, then $\eta \in R \cap T$ for some $T \in \mathcal{F}$. The $\eta \in R$ and $\eta \in T$ where $T \in \mathcal{F}$. Then $\eta \in R$ and $\eta \in T$, so that $\eta \in \bigcup_{S \in \mathcal{F}} S$. Therefore $\eta \in R \cap (\bigcup_{S \in \mathcal{F}} S)$.

Hence $R \cap (\bigcup_{S \in \mathcal{F}} S) = \bigcup_{S \in \mathcal{F}} (R \cap S)$, as claimed. □

Lemma II.1.2

Lemma II.1.2. DeMorgan’s Laws

If $\mathcal{F}$ is a family of subsets of a set $X$, and if for any given set $S$ we denote by $S' = X \setminus S$ the complement of $S$ with respect to $X$, then

$$(\bigcup_{S \in \mathcal{F}} S)' = \bigcap_{S \in \mathcal{F}} S' \quad \text{and} \quad (\bigcap_{S \in \mathcal{F}} S)' = \bigcup_{S \in \mathcal{F}} S'.$$

Proof. To establish the first claim, let $\xi \in (\bigcup_{S \in \mathcal{F}} S)'$. Then $\xi \notin \bigcup_{S \in \mathcal{F}} S$ and so $\xi \notin S$ for all $S \in \mathcal{F}$. That is, $\xi \in S'$ for all $S \in \mathcal{F}$ and so $\xi \in \bigcap_{S \in \mathcal{F}} S'$. Hence $(\bigcup_{S \in \mathcal{F}} S)' \subseteq \bigcap_{S \in \mathcal{F}} S'$.

Conversely, if $\eta \in \bigcap_{S \in \mathcal{F}} S'$ then $\eta \in S'$ for all $S \in \mathcal{F}$. That is, $\eta \notin S$ for all $S \in \mathcal{F}$. Therefore $\eta \notin \bigcup_{S \in \mathcal{F}} S$ and so $\eta \notin (\bigcup_{S \in \mathcal{F}} S)'$. Hence $\bigcap_{S \in \mathcal{F}} S' \supseteq (\bigcup_{S \in \mathcal{F}} S)'$ and so $(\bigcup_{S \in \mathcal{F}} S)' = \bigcap_{S \in \mathcal{F}} S'$, as claimed.

Lemma II.1.2 (continued)

Lemma II.1.2. DeMorgan’s Laws.

If $\mathcal{F}$ is a family of subsets of a set $X$, and if for any given set $S$ we denote by $S' = X \setminus S$ the complement of $S$ with respect to $X$, then

$$(\bigcup_{S \in \mathcal{F}} S)' = \bigcap_{S \in \mathcal{F}} S' \quad \text{and} \quad (\bigcap_{S \in \mathcal{F}} S)' = \bigcup_{S \in \mathcal{F}} S'.$$

Proof (continued). We can take a short cut to prove the second claim. Define $\mathcal{F}' = \{S' \mid S \in \mathcal{F}\}$. Then applying the first claim to family $\mathcal{F}'$ we have $(\bigcup_{S \in \mathcal{F}} S)' = \bigcap_{S \in \mathcal{F}} S'$ and taking complements of both sides

$$(\bigcup_{S \in \mathcal{F}} S)' = (\bigcap_{S \in \mathcal{F}} S)'$$

(since $(R)' = R'' = R$ for any set $R$). Replacing $S'$ with $S$ (and $\mathcal{F}$ with $\mathcal{F}'$) gives $\bigcup_{S \in \mathcal{F}} S = (\bigcap_{S \in \mathcal{F}} S)'$, as claimed. □
Theorem II.1.1

Theorem II.1.1. If the class $\mathcal{H}$ of subsets of a set $\mathcal{X}$ is a Boolean algebra, then

(a) the entire set $\mathcal{X}$ and the empty set $\emptyset$ belong to $\mathcal{H}$,
(b) the intersection $R \cap S$ belongs to $\mathcal{H}$ whenever $R, S \in \mathcal{H}$, and
(c) the difference $R \setminus S$ and symmetric difference

\[ R \triangle S = (R \setminus S) \cup (S \setminus R) \]

belong to $\mathcal{H}$ whenever $R, S \in \mathcal{H}$.

Proof. 1. If $R \subseteq \mathcal{X}$ and $R \in \mathcal{H}$, then $R' \in \mathcal{H}$ and so

\[ R \cup R' = \mathcal{X} \in \mathcal{H}, \]

as claimed. Then $\mathcal{X} = \emptyset \in \mathcal{H}$, as claimed.

2. For $R, S \in \mathcal{H}$ we have $R \cup S \in \mathcal{H}$ and $(R \cup S)' = R' \cap S' \in \mathcal{H}$ (by Lemma II.1.1), as claimed.

Proof (continued). 3. For $R, S \in \mathcal{H}$ we have

\[ R \setminus S = R \cap S' \]

\[ = (R' \cup S')' \]

by Lemma II.1.2

\[ = (R' \cup S') \in \mathcal{H} \]

since $\mathcal{H}$ is a Boolean algebra.

Similarly, $S \setminus R \in \mathcal{H}$. Hence $R \triangle S = (R \setminus S) \cup (S \setminus R) \in \mathcal{H}$, as claimed.

Theorem II.1.2

Theorem II.1.2. For any given nonempty family $\mathcal{F}$ of subset of a set $\mathcal{X}$, there is a unique smallest Boolean algebra $\mathcal{A}(\mathcal{F})$ and a unique smallest Boolean $\sigma$ algebra $\mathcal{A}_\sigma(\mathcal{F})$ containing $\mathcal{F}$. That is, if $\mathcal{A}$ is a Boolean algebra containing $\mathcal{F}$ then $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{A}$ and if $\mathcal{A}_\sigma$ is a Boolean algebra containing $\mathcal{F}$ then $\mathcal{A}_\sigma(\mathcal{F}) \subseteq \mathcal{A}_\sigma$. $\mathcal{A}(\mathcal{F})$ and $\mathcal{A}_\sigma(\mathcal{F})$ are called, respectively, the Boolean algebra and the Boolean $\sigma$ algebra generated by the family $\mathcal{F}$.

Proof. Denote by $\mathcal{F}$ the family of all Boolean algebras $\mathcal{A}$ containing $\mathcal{F}$. $\mathcal{F}$ is not empty because it contains the power set $\mathcal{P}(\mathcal{X})$ of all subsets of $\mathcal{X}$. Consider the family $\mathcal{A}(\mathcal{F}) = \cap \mathcal{A} \in \mathcal{F}$. Now $\mathcal{A}(\mathcal{F})$ is nonempty since $\mathcal{F} \subseteq \mathcal{A}(\mathcal{F})$. If $R, S \in \mathcal{A}(\mathcal{F})$ then $R, S \in \mathcal{A}$ for all $\mathcal{A} \in \mathcal{F}$ and since each $\mathcal{A}$ is an algebra then $R \cup S \in \mathcal{A}$ and $R' = \mathcal{X} \setminus R \in \mathcal{A}$ for all $\mathcal{A} \in \mathcal{F}$. Therefore $R \cup S \in \mathcal{A}(\mathcal{F})$, $R' \in \mathcal{A}(\mathcal{F})$, and $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{F}$. If $\mathcal{A}$ is any Boolean algebra containing $\mathcal{F}$ then $\mathcal{A} \in \mathcal{F}$ and so $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{A}$; that is, $\mathcal{A}(\mathcal{F})$ is a smallest Boolean algebra containing $\mathcal{F}$, as claimed.

Proof (continued). For uniqueness, if $\mathcal{A}_1(\mathcal{F})$ and $\mathcal{A}_2(\mathcal{F})$ are two such algebras, then $\mathcal{A}_1(\mathcal{F}) \subseteq \mathcal{A}_2(\mathcal{F})$ since $\mathcal{A}_1(\mathcal{F})$ is a smallest algebra and $\mathcal{A}_2(\mathcal{F}) \subseteq \mathcal{A}_2(\mathcal{F})$ since $\mathcal{A}_2(\mathcal{F})$ is a smallest algebra. So $\mathcal{A}_1(\mathcal{F}) = \mathcal{A}_2(\mathcal{F})$ and the smallest such algebra is unique.

The proof for a smallest Boolean $\sigma$ algebra is similar.
Theorem II.1.3. The family $\mathcal{B}_0^n$ of all finite unions

$$\bigcup_{i=1}^k I_i$$

where $I_1, I_2, \ldots, I_k \in \mathcal{I}^n$ and $k \in \mathbb{N}$

of intervals in $\mathcal{I}^n$ is identical to the Boolean algebra $\mathcal{A}(\mathcal{I}^n)$.

Proof. Since a Boolean algebra is closed under finite unions, then $\mathcal{B}_0^n \subset \mathcal{A}(\mathcal{I}^n)$. Now if we show that $\mathcal{B}_0^n$ is a Boolean algebra then we must have $\mathcal{A}(\mathcal{I}^n) \subset \mathcal{B}_0^n$ (since $\mathcal{A}(\mathcal{I}^n)$ is the smallest Boolean algebra containing $\mathcal{I}^n$) and hence $\mathcal{B}_0^n = \mathcal{A}(\mathcal{I}^n)$. If $R, S \in \mathcal{B}_0^n$ then $R = \bigcup_{i=1}^k I_i$ and $S = \bigcup_{j=1}^l J_j$ for some $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_l \in \mathcal{B}_0^n$.

To prove $R' \in \mathcal{B}_0^n$, we proceed by induction on $k$ where $R' = \bigcup_{i=1}^{k+1} I_i$ and $I_k$ is an interval. In the case $k = 1$ we have $R' = I_1$ is an interval. In Exercise II.1A it is to be shown that $I_1' = I_1^{(1)} \cup I_1^{(2)} \cup \cdots \cup I_1^{(v)}$ where $I_1^{(1)}, I_1^{(2)}, \ldots, I_1^{(v)} \in \mathcal{I}^n$ and $v \leq 2^3 3^{n-1}$. So $R' \in \mathcal{B}_0^n$ and $\mathcal{B}_0^n$ is closed under complements of intervals.

Theorem II.1.3 (continued 2)

Theorem II.1.3. The family $\mathcal{B}_0^n$ of all finite unions

$$\bigcup_{i=1}^k I_i$$

where $I_1, I_2, \ldots, I_k \in \mathcal{I}^n$ and $k \in \mathbb{N}$

of intervals in $\mathcal{I}^n$ is identical to the Boolean algebra $\mathcal{A}(\mathcal{I}^n)$.

Proof (continued). Now an intersection of two elements of $\mathcal{I}^n$ is an element of $\mathcal{I}^n$, so $J_m \cap J^{(1)}, J_m \cap J^{(2)}, \ldots, J_m \cap J^{(v)} \in \mathcal{I}^n$ and so $R' = \bigcup_{i=1}^{k-1} I_i' \cup I_{k+1}' \in \mathcal{B}_0^n$ and so by Mathematical Induction $\mathcal{B}_0^n$ is closed under complements. Therefore $\mathcal{B}_0^n$ is a Boolean algebra and the claim holds, as explained above.

Theorem II.1.4

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof. Assume $O$ is an open set in $\mathbb{R}^n$. For each $m \in \mathbb{N}$ consider the open intervals in $\mathbb{R}^n$

$$I_{k_1, k_2, \ldots, k_n}^{(m)} = \left\{ x = (x_1, x_2, \ldots, x_n) \left| \frac{k_1 - 1}{m} < x_1 < \frac{k_1 + 1}{m}, \ldots, \frac{k_n - 1}{m} < x_n < \frac{k_n + 1}{m} \right. \right\}$$

for $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Then this countable collection of intervals covers $\mathbb{R}^n$. Consider the collection of all such intervals lying within $O$:

$$R^{(m)} = \left\{ I_{k_1, k_2, \ldots, k_n}^{(m)} \subseteq O \text{ where } k_1, k_2, \ldots, k_n \in \mathbb{N} \right\}$$

Then $R^{(m)}$ is countable for each $m \in \mathbb{Z}$. 
Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof (continued). Now if $x \in O$ then there is an $\varepsilon$-neighborhood of $x$ contained in $O$ (using the Euclidean metric on $\mathbb{R}^n$ to define such a neighborhood) and so for $m$ sufficiently large (namely, $m > 2n/\varepsilon$) there is an interval $I_{k_1,k_2,\ldots,k_n}^{(m)}$ containing $x$ and lying in the $\varepsilon$ neighborhood. Now let $A$ be the union of the intervals in the $\mathbb{R}^{m}$, $A = \bigcup_{m \in \mathbb{Z}, I \in \mathbb{R}^{m}} I$. Since each element of each $R^{(m)}$ is a subset of $O$, then $A \subseteq O$. Since each $x \in O$ is in some element of some $R^{(m)}$ then $O \subseteq A$. So $O = A$ and $O$ is a countable union of intervals. Since the Borel sets are in the $\sigma$ algebra generated by $\mathcal{A}^{n}$, then $O$ is a Borel set. Since any closed set $C$ has an open complement and a $\sigma$ algebra is closed under complements, then each closed set in $\mathbb{R}^n$ is also Borel. □

Theorem II.1.6. Every Boolean $\sigma$ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof. (a) If $\mathcal{A}_\sigma$ is a Boolean $\sigma$ algebra and $R_1, R_2, \ldots \in \mathcal{A}_\sigma$ is monotonically increasing then $\lim_{k \to \infty} R_k = \sup_{k=1}^{\infty} R_k \in \mathcal{A}_\sigma$. In the case that $S_1, S_2, \ldots$ is a monotonically decreasing sequence in $\mathcal{A}_\sigma$ and so $\lim_{k \to \infty} (S_1 \setminus S_k) = \bigcap_{k=1}^{\infty} (S_1 \setminus S_k) \in \mathcal{A}_\sigma$, where by Lemmas II.1.1 and II.1.2,

$$\bigcap_{k=1}^{\infty} (S_1 \setminus S_k) = \bigcap_{k=1}^{\infty} (S_1 \cap S_k^c) = S_1 \cap \left( \bigcap_{k=1}^{\infty} S_k^c \right) = S_1 \setminus \lim_{k \to \infty} S_k.$$

So $\lim_{k \to \infty} S_k \in \mathcal{A}_\sigma$ and since $\mathcal{A}_\sigma$ is closed under set differences (and $S_1$ contains $S_2, S_3, \ldots$) then $\lim_{k \to \infty} S_k = \lim_{k \to \infty} S_k \in \mathcal{A}_\sigma$. So $\mathcal{A}_\sigma$ is a monotone class, as claimed.

Theorem II.1.7. If $\mathcal{A}$ is a Boolean algebra and $\mathcal{M}(\mathcal{A})$ is the monotone class generated by $\mathcal{A}$, then $\mathcal{M}(\mathcal{A})$ is identical with the Boolean $\sigma$ algebra $\mathcal{A}_\sigma(\mathcal{A})$ generated by the family $\mathcal{A}$ of sets.

Proof. By Theorem II.1.6, $\mathcal{A}_\sigma(\mathcal{A})$ is a monotone class and by definition $\mathcal{M}(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$, so $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{A}_\sigma(\mathcal{A})$. We will show that $\mathcal{M}(\mathcal{A})$ is a Boolean $\sigma$ algebra containing $\mathcal{A}$. Since $\mathcal{A}_\sigma(\mathcal{A})$ is the smallest $\sigma$ algebra containing $\mathcal{A}$, then this will imply $\mathcal{A}_\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ and hence $\mathcal{A}_\sigma(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

For $R \in \mathcal{M}(\mathcal{A})$, denote by $\mathcal{N}(R)$ the family of sets $S \in \mathcal{M}(\mathcal{A})$ such that $S^c \cup R \in \mathcal{M}(\mathcal{A})$. So by definition, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. If $S_1, S_2, \ldots \in \mathcal{M}(R)$ is a monotone sequence then, since $\mathcal{M}(\mathcal{A})$ is a monotone class, by Exercise II.1.6 ($\lim_{n \to \infty} S_n)^c \subseteq \lim_{n \to \infty} S_n^c \in \mathcal{M}(\mathcal{A})$, and the sequence of sets $R \cup S_1, R \cup S_2, \ldots \in \mathcal{M}(\mathcal{A})$ is also a monotone sequence and, again by Exercise II.1.6, $(\lim_{n \to \infty} S_n) \cup R = \lim_{n \to \infty} (S_n \cup R) \in \mathcal{M}(\mathcal{A})$. So, by the definition of $\mathcal{M}(R)$, $\lim_{n \to \infty} S_n \in \mathcal{M}(R)$ and so $\mathcal{M}(R)$ is a monotone class.
Theorem II.1.7 (continued 1)

**Proof (continued).** In addition, if $R \in \mathcal{A}$ then $\mathcal{M}(R)$ is not empty because it contains $\mathcal{A}$, since $\mathcal{A}$ is a Boolean algebra. In this case, $\mathcal{M}(R)$ is a monotone class containing $\mathcal{A}$ ad hence $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(R)$. Therefore $\mathcal{M}(R) = \mathcal{M}(\mathcal{A})$ for $R \in \mathcal{A}$. Since $\mathcal{A}$ is a Boolean algebra of sets then $R^c = \mathcal{M}(\mathcal{A})$. By the definition of $\mathcal{M}(\mathcal{A})$, for every $S \in \mathcal{M}(R^c) = \mathcal{M}(\mathcal{A})$ we have $S^c \in \mathcal{M}(\mathcal{A})$ so that $\mathcal{M}(\mathcal{A})$ is closed under complements.

Furthermore, if $R \in \mathcal{A}$, then for an arbitrary $S \in \mathcal{M}(\mathcal{A})$ we have $S \in \mathcal{M}(R) = \mathcal{M}(\mathcal{A})$, i.e., $R \cup S \in \mathcal{M}(\mathcal{A})$; consequently (by the definition of $\mathcal{M}(S)$), $R \in \mathcal{M}(S)$. Therefore $\mathcal{A} \subseteq \mathcal{M}(S)$. As shown above, $\mathcal{M}(S)$ is a monotone class and, since it contains $\mathcal{A}$, we have $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(S)$. Since $\mathcal{M}(S) \subseteq \mathcal{M}(\mathcal{A})$ by definition, then $\mathcal{M}(S) = \mathcal{M}(\mathcal{A})$ where $S$ is any set in $\mathcal{M}(\mathcal{A})$. So for any $R, S \in \mathcal{M}(S) = \mathcal{M}(\mathcal{A})$ we have (by the definition of $\mathcal{M}(S)$) that $R \cup S, R' \in \mathcal{M}(S) = \mathcal{M}(\mathcal{A})$. Therefore, $\mathcal{M}(\mathcal{A})$ is a Boolean algebra.

Theorem II.1.7 (continued 2)

**Theorem II.1.7.** If $\mathcal{A}$ is a Boolean algebra and $\mathcal{M}(\mathcal{A})$ is the monotone class generated by $\mathcal{A}$, then $\mathcal{M}(\mathcal{A})$ is identical with the Boolean $\sigma$ algebra $\mathcal{A}_\sigma(\mathcal{A})$ generated by the family $\mathcal{A}$ of sets.

**Proof (continued).** Since $\mathcal{M}(\mathcal{A})$ is a Boolean algebra which is a monotone class then, by Theorem II.1.6, $\mathcal{M}(\mathcal{A})$ is a Boolean $\sigma$ algebra. The result now follows, as explained above. $\square$