Modern Algebra

Chapter II. Measure Theory and Hilbert Spaces of Functions

II.1. Measurable Spaces—Proofs of Theorems

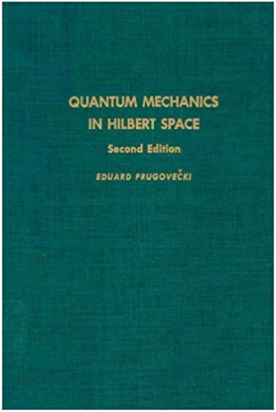

Table of contents

1. Lemma II.1.1
2. Lemma II.1.2
3. Theorem II.1.1
4. Theorem II.1.2
5. Theorem II.1.3
6. Theorem II.1.4
7. Theorem II.1.6
8. Theorem II.1.7

Lemma II.1.1

Lemma II.1.1. If $\mathscr{F}$ is a family of sets and R is any given set, then

$$R \cap (\cup_{S \in \mathscr{F}} S) = \cup_{S \in \mathscr{F}} (R \cap S).$$

Proof. If $\xi \in R \cap (\cup_{S \in \mathscr{F}} S)$ then $\xi \in R$ and $\xi \in \cup_{S \in \mathscr{F}} S$. That is, $\xi \in T$ for some $T \in \mathscr{F}$. Then $\xi \in R \cap T$ where $T \in \mathscr{F}$ and so $\xi \in \cup_{S \in \mathscr{F}} (R \cap S)$.

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Conversely, if $\eta \in \cup_{S \in \mathscr{F}} (R \cap S)$, then $\eta \in R \cap T$ for some $T \in \mathscr{F}$. The $\eta \in R$ and $\eta \in T$ where $T \in \mathscr{F}$. Then $\eta \in R$ and $\eta \in T$, so that $\eta \in \cup_{S \in \mathscr{F}} S$. Therefore $\eta \in R \cap (\cup_{S \in \mathscr{F}} S)$.

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Hence $R \cap (\cup_{S \in \mathscr{F}} S) = \cup_{S \in \mathscr{F}} (R \cap S)$, as claimed. □

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Lemma II.1.2. DeMorgan's Laws

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If $\mathscr{F}$ is a family of subsets of a set $\mathscr{X}$, and if for any given set S we denote by $S' = \mathscr{X} \setminus S$ the *complement* of S with respect to $\mathscr{X}$, then

$$(\cup_{S\in\mathscr{F}} S)' = \cap_{S\in\mathscr{F}} S' \text{ and } (\cap_{S\in\mathscr{F}} S)' = \cup_{S\in\mathscr{F}} S'.$$

Proof. To establish the first claim, let $\xi \in (\cup_{S\in\mathscr{F}} S)'$. Then $\xi \notin \cup_{S\in\mathscr{F}} S$ and so $\xi \notin S$ for all $S \in \mathscr{F}$. That is, $\xi \in S'$ for all $S \in \mathscr{F}$ and so $\xi \in \cap_{S\in\mathscr{F}} S'$. Hence $(\cup_{S\in\mathscr{F}} S)' \subset \cap_{S\in\mathscr{F}} S'$.

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Conversely, if $\eta \in \cap_{S\in\mathscr{F}} S'$ then $\eta \in S'$ for all $S \in \mathscr{F}$. That is, $\eta \notin S$ for all $S \in \mathscr{F}$. Therefore $\eta \notin \cup_{S\in\mathscr{F}} S$ and so $\eta \in (\cup_{S\in\mathscr{F}} S)'$. Hence $\cap_{S\in\mathscr{F}} S' \subset (\cup_{S\in\mathscr{F}} S)'$ and so $(\cup_{S\in\mathscr{F}} S)' = \cap_{S\in\mathscr{F}} S'$, as claimed.

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Conversely, if $\eta \in \cap_{S\in\mathscr{F}} S'$ then $\eta \in S'$ for all $S \in \mathscr{F}$. That is, $\eta \notin S$ for all $S \in \mathscr{F}$. Therefore $\eta \notin \cup_{S\in\mathscr{F}} S$ and so $\eta \in (\cup_{S\in\mathscr{F}} S)'$. Hence $\cap_{S\in\mathscr{F}} S' \subset (\cup_{S\in\mathscr{F}} S)'$ and so $(\cup_{S\in\mathscr{F}} S)' = \cap_{S\in\mathscr{F}} S'$, as claimed.

Lemma II.1.2 (continued)

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If $\mathscr{F}$ is a family of subsets of a set $\mathscr{X}$, and if for any given set S we denote by $S' = \mathscr{X} \setminus S$ the *complement* of S with respect to $\mathscr{X}$, then

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Proof (continued). We can take a short cut to prove the second claim. Define $\mathscr{F}' = \{S' \mid S \in \mathscr{F}\}$. Then applying the first claim to family $\mathscr{F}'$ we have $(\cup_{S' \in \mathscr{F}'} S')' = \cap_{S' \in \mathscr{F}'} S'$ and taking complements of both sides

$$\left((\cup_{S' \in \mathscr{F}'} S')' \right)' = (\cap_{S' \in \mathscr{F}'} S')' \text{ or } \cup_{S' \in \mathscr{F}'} S' = (\cap_{S' \in \mathscr{F}'} S')'$$

(since $(R')' = R'' = R$ for any set R). Replacing S' with S (and $\mathscr{F}$ with $\mathscr{F}$) gives $\cup_{S \in \mathscr{F}} S = (\cap_{S \in \mathscr{F}} S)'$, as claimed. □

Theorem II.1.1

Theorem II.1.1. If the class $\mathscr{K}$ of subsets of a set $\mathscr{X}$ is a Boolean algebra, then

(a) the entire set $\mathscr{X}$ and the empty set $\varnothing$ belong to $\mathscr{K}$,

(b) the intersection $R \cap S$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$, and

(c) the difference $R \setminus S$ and symmetric difference $R \triangle S = (R \setminus S) \cup (S \setminus R)$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$.

Proof. 1. If $R \subset \mathscr{X}$ and $R \in \mathscr{K}$, then $R' \in \mathscr{K}$ and so $R \cup R' = \mathscr{X} \in \mathscr{K}$, as claimed. Then $\mathscr{X}' = \varnothing \in \mathscr{K}$, as claimed.

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2. For $R, S \in \mathscr{K}$ we have $R \cup S \in \mathscr{K}$ and $(R \cup S)' = R' \cap S' \in \mathscr{K}$ (by Lemma II.1.1), as claimed.

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Proof (continued). 3. For $R, S \in \mathscr{K}$ we have

$$\begin{aligned} R \setminus S &= R \cap S' \\ &= (R' \cup S'')' \text{ by Lemma II.1.2} \\ &= (R' \cup S)' \\ &\in \mathscr{K} \text{ since } \mathscr{K} \text{ is a Boolean algebra.} \end{aligned}$$

Similarly, $S \setminus R \in \mathscr{K}$. Hence $R \triangle S = (R \setminus S) \cup (S \setminus R) \in \mathscr{K}$, as claimed. □

Theorem II.1.2

Theorem II.1.2. For any given nonempty family $\mathscr{F}$ of subset of a set $\mathscr{X}$ there is a unique smallest Boolean algebra $\mathscr{A}(\mathscr{F})$ and a unique smallest Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{F})$ containing $\mathscr{F}$. That is, if $\mathscr{A}$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$ and if $\mathscr{A}_\sigma$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}_\sigma(\mathscr{F}) \subset \mathscr{A}_\sigma$. $\mathscr{A}(\mathscr{F})$ and $\mathscr{A}_\sigma(\mathscr{F})$ are called, respectively, the *Boolean algebra* and the *Boolean σ algebra generated* by the family $\mathscr{F}$.

Proof. Denote by $\mathfrak{F}$ the family of all Boolean algebras $\mathscr{A}$ containing $\mathscr{F}$. $\mathfrak{F}$ is not empty because it contains the power set $\mathfrak{S}_{\mathscr{X}}$ of all subsets of $\mathscr{X}$. Consider the family $\mathscr{A}(\mathscr{F}) = \cap_{\mathfrak{S}\in\mathfrak{F}}\mathfrak{S}$. Now $\mathscr{A}(\mathscr{F})$ is nonempty since $\mathscr{F} \subset \mathscr{A}(\mathscr{F})$.

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Theorem II.1.2 (continued)

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Proof (continued). For uniqueness, if $\mathscr{A}_1(\mathscr{F})$ and $\mathscr{A}_2(\mathscr{F})$ are two such algebras, then $\mathscr{A}_1(\mathscr{F}) \subset \mathscr{A}_2(\mathscr{F})$ since $\mathscr{A}_1(\mathscr{F})$ is a smallest algebra and $\mathscr{A}_2(\mathscr{F}) \subset \mathscr{A}_1(\mathscr{F})$ since $\mathscr{A}_2(\mathscr{F})$ is a smallest algebra. So $\mathscr{A}_1(\mathscr{F}) = \mathscr{A}_2(\mathscr{F})$ and the smallest such algebra is unique.

The proof for a smallest Boolean σ algebra is similar. □

Theorem II.1.2 (continued)

Theorem II.1.2. For any given nonempty family $\mathscr{F}$ of subset of a set $\mathscr{X}$ there is a unique smallest Boolean algebra $\mathscr{A}(\mathscr{F})$ and a unique smallest Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{F})$ containing $\mathscr{F}$. That is, if $\mathscr{A}$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$ and if $\mathscr{A}_\sigma$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}_\sigma(\mathscr{F}) \subset \mathscr{A}_\sigma$. $\mathscr{A}(\mathscr{F})$ and $\mathscr{A}_\sigma(\mathscr{F})$ are called, respectively, the *Boolean algebra* and the *Boolean σ algebra generated* by the family $\mathscr{F}$.

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Theorem II.1.3

Theorem II.1.3. The family $\mathscr{B}_0^n$ of all finite unions

$$I_1 \cup I_2 \cup \cdots \cup I_k \text{ where } I_1, I_2, \ldots, I_n \in \mathscr{I}^n \text{ and } k \in \mathbb{N}$$

of intervals in $\mathscr{I}^n$ is identical to the Boolean algebra $\mathscr{A}(\mathscr{I}^n)$.

Proof. Since a Boolean algebra is closed under finite unions, then $\mathscr{B}_0^n \subset \mathscr{A}(\mathscr{I}^n)$. Now if we show that $\mathscr{B}_0^n$ is a Boolean algebra then we must have $\mathscr{A}(\mathscr{I}^n) \subset \mathscr{B}_0^n$ (since $\mathscr{A}(\mathscr{I}^n)$ is the smallest Boolean algebra containing $\mathscr{I}^n$) and hence $\mathscr{B}_0^n = \mathscr{A}(\mathscr{I}^n)$. If $R, S \in \mathscr{B}_0^n$ then $R = I_1 \cup I_2 \cup \cdots \cup I_k$ and $S = J_1 \cup J_2 \cup \cdots \cup J_\ell$ for some $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_\ell \in \mathscr{B}_0^n$.

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To prove $R' \in \mathscr{B}_0^n$, we proceed by induction on k where $R = I_1 \cup I_2 \cup \cdots \cup I_k$. In the case $k = 1$ we have $R = I_1$ is an interval. In Exercise II.1.A it is to be shown that $I_1' = I^{(1)} \cup I^{(2)} \cup \cdots \cup I^{(v)}$ where $I^{(1)}, I^{(2)}, \ldots, I^{(v)} \in \mathscr{I}^n$ and $v \leq 2^n 3^{n-1}$. So $R' \in \mathscr{B}_0^n$ and $\mathscr{B}_0^n$ is closed under complements of intervals.

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Proof. Since a Boolean algebra is closed under finite unions, then $\mathscr{B}_0^n \subset \mathscr{A}(\mathscr{I}^n)$. Now if we show that $\mathscr{B}_0^n$ is a Boolean algebra then we must have $\mathscr{A}(\mathscr{I}^n) \subset \mathscr{B}_0^n$ (since $\mathscr{A}(\mathscr{I}^n)$ is the smallest Boolean algebra containing $\mathscr{I}^n$) and hence $\mathscr{B}_0^n = \mathscr{A}(\mathscr{I}^n)$. If $R, S \in \mathscr{B}_0^n$ then $R = I_1 \cup I_2 \cup \cdots \cup I_k$ and $S = J_1 \cup J_2 \cup \cdots \cup J_\ell$ for some $I_1, I_2, \ldots, I_k, J_1, J_2, \ldots, J_\ell \in \mathscr{B}_0^n$.

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Theorem II.1.3 (continued 1)

Proof (continued). Suppose $R' \in \mathscr{B}_0^n$ for all $R = I_1 \cup I_2 \cup \cdots \cup I_k$ for intervals $I_1, I_2, \ldots, I_k \in \mathscr{I}^n$. Then

$$R' = (I_1 \cup I_2 \cup \cdots \cup I_k)' = \cup_{m=1}^{p} J_m \qquad (1.5)$$

for some $p \in \mathbb{N}$ and $J_1, J_2, \ldots, J_p \in \mathscr{I}^n$ (since the result holds for $k = 1$ instead), and we have

$$(I_1 \cup I_2 \cup \cdots \cup I_{k+1})'$$

$$\begin{aligned}
&= (I_1 \cup I_2 \cup \cdots \cup I_k)' \cap I_{k+1} \text{ by Lemma II.1.2} \\
&= \left(\cup_{m=1}^{p} J_m\right) \cap \left(J^{(1)} \cup J^{(2)} \cup \cdots \cup J^{(v)}\right) \text{ by (1.5)} \\
&= \left(\left(\cup_{m=1}^{p} J_m\right) \cap J^{(1)}\right) \cup \left(\left(\cup_{m=1}^{p} J_m\right) \cap J^{(2)}\right) \cup \cdots \cup \left(\left(\cup_{m=1}^{p} J_m\right) \cap J^{(v)}\right) \\
&\quad \text{by Lemma II.1.1} \\
&= \left(\cup_{m=1}^{p} J_m \cap J^{(1)}\right) \cup \left(\cup_{m=1}^{p} J_m \cap J^{(2)}\right) \cup \cdots \cup \left(\cup_{m=1}^{p} J_m \cap J^{(v)}\right) \\
&\quad \text{by Lemma II.1.1.}
\end{aligned}$$

Theorem II.1.3 (continued 2)

Theorem II.1.3. The family $\mathscr{B}_0^n$ of all finite unions

$$I_1 \cup I_2 \cup \cdots \cup I_k \text{ where } I_1, I_2, \ldots, I_n \in \mathscr{I}^n \text{ and } k \in \mathbb{N}$$

of intervals in $\mathscr{I}^n$ is identical to the Boolean algebra $\mathscr{A}(\mathscr{I}^n)$.

Proof (continued). Now an intersection of two elements of $\mathscr{I}^n$ is an element of $\mathscr{I}^n$, so $J_m \cap J^{(1)}, J_m \cap J^{(2)}, \ldots, J_m \cap J^{(v)} \in \mathscr{I}^n$ and so $R' = (I_1 \cup I_2 \cup \cdots \cup I_{k+1}) \in \mathscr{B}_0^n$ and so by Mathematical Induction $\mathscr{B}_0^n$ is closed under complements. Therefore $\mathscr{B}_0^n$ is a Boolean algebra and the claim holds, as explained above. □

Theorem II.1.4

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof. Assume O is an open set in $\mathbb{R}^n$. For each $m \in \mathbb{N}$ consider the open intervals in $\mathbb{R}^n$

$$I^{(m)}_{k_1,k_2,\ldots,k_n} = \left\{ x = (x_1, x_2, \ldots, x_n) \,\middle|\, \frac{k_1 - 1}{m} < x_1 < \frac{k_1 + 1}{m}, \right.$$

$$\left. \frac{k_2 - 1}{m} < x_2 < \frac{k_2 + 1}{m}, \ldots, \frac{k_n - 1}{m} < x_n < \frac{k_n + 1}{m} \right\}$$

$$= \left[\frac{k_1 - 1}{m}, \frac{k_1 + 1}{m}\right] \times \left[\frac{k_2 - 1}{m}, \frac{k_2 + 1}{m}\right] \times \cdots \times \left[\frac{k_n - 1}{m}, \frac{k_n + 1}{m}\right]$$

for $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Then this countable collection of intervals covers $\mathbb{R}^n$. Consider the collection of all such intervals lying within O:

$$R^{(m)} = \{ I^{(m)}_{k_1,k_2,\ldots,k_n} \mid I^{(m)}_{k_1,k_2,\ldots,k_n} \subset O \text{ where } k_1, k_2, \ldots, k_n \in \mathbb{N} \}.$$

Then $R^{(m)}$ is countable for each $m \in \mathbb{Z}$.

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$$\left. \frac{k_2 - 1}{m} < x_2 < \frac{k_2 + 1}{m}, \ldots, \frac{k_n - 1}{m} < x_n < \frac{k_n + 1}{m} \right\}$$

$$= \left[\frac{k_1 - 1}{m}, \frac{k_1 + 1}{m} \right] \times \left[\frac{k_2 - 1}{m}, \frac{k_2 + 1}{m} \right] \times \cdots \times \left[\frac{k_n - 1}{m}, \frac{k_n + 1}{m} \right]$$

for $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Then this countable collection of intervals covers $\mathbb{R}^n$. Consider the collection of all such intervals lying within O:

$$R^{(m)} = \{ I^{(m)}_{k_1,k_2,\ldots,k_n} \mid I^{(m)}_{k_1,k_2,\ldots,k_n} \subset O \text{ where } k_1, k_2, \ldots, k_n \in \mathbb{N} \}.$$

Then $R^{(m)}$ is countable for each $m \in \mathbb{Z}$.

Theorem II.1.4 (continued)

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Proof (continued). Now if $x \in O$ then there is an ε-neighborhood of x contained in O (using the Euclidean metric on $\mathbb{R}^n$ to define such a neighborhood) and so for m sufficiently large (namely, $m > 2n/\varepsilon$) there is an interval $I^{(m)}_{k_1,k_2,\ldots,k_n}$ containing x and lying in the ε neighborhood. Now let A be the union of the intervals in the $\mathbb{R}^{(m)}$, $A = \cup_{m\in\mathbb{Z}, I\in R^{(m)}} I$. Since each element of each $R^{(m)}$ is a subset of O, then $A \subset O$. Since each $x \in O$ is in some element of some $R^{(m)}$ then $O \subset A$. So $O = A$ and O is a countable union of intervals. Since the Borel sets are in the σ algebra generated by $\mathscr{I}^n$, then O is a Borel set. Since any closed set C has an open complement and a σ algebra is closed under complements, then each closed set in $\mathbb{R}^n$ is also Borel. □

Theorem II.1.4 (continued)

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

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Theorem II.1.6

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof. (a) If $\mathscr{A}_\sigma$ is a Boolean σ algebra and $R_1, R_2, \ldots \in \mathscr{A}_\sigma$ is monotonically increasing then $\lim_{k\to\infty} R_k = \sup_{k=1}^\infty R_k \in \mathscr{A}_\sigma$. In the case that $S_1, S_2, \ldots$ is a monotonically decreasing sequence in $\mathscr{A}_\sigma$ and so $\lim_{k\to\infty}(S_1 \setminus S_k) = \cup_{k=1}^\infty (S_1 \setminus S_k) \in \mathscr{A}_\sigma$, where by Lemmas II.1.1 and II.1.2,

$$\cup_{k=1}^\infty (S_1 \setminus S_k) = \cup_{k=1}^\infty (S_1 \cap S_k') = S_1 \cap \left(\cup_{k=1}^\infty S_k'\right) S_1 \setminus \left(\cup_{k=1}^\infty S_k\right)'$$

$$= S_1 \setminus \left(\cap_{k=1}^\infty S_k\right) = S_1 \setminus \lim_{k\to\infty} S_k.$$

Theorem II.1.6

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof. (a) If $\mathscr{A}_\sigma$ is a Boolean σ algebra and $R_1, R_2, \ldots \in \mathscr{A}_\sigma$ is monotonically increasing then $\lim_{k\to\infty} R_k = \sup_{k=1}^{\infty} R_k \in \mathscr{A}_\sigma$. In the case that $S_1, S_2, \ldots$ is a monotonically decreasing sequence in $\mathscr{A}_\sigma$ and so $\lim_{k\to\infty}(S_1 \setminus S_k) = \cup_{k=1}^{\infty}(S_1 \setminus S_k) \in \mathscr{A}_\sigma$, where by Lemmas II.1.1 and II.1.2,

$$\cup_{k=1}^{\infty}(S_1 \setminus S_k) = \cup_{k=1}^{\infty}(S_1 \cap S_k') = S_1 \cap \left(\cup_{k=1}^{\infty} S_k'\right) S_1 \setminus (\cup_{k=1}^{\infty} S_k)'$$

$$= S_1 \setminus (\cap_{k=1}^{\infty} S_k) = S_1 \setminus \lim_{k\to\infty} S_k.$$

So $S_1 \setminus \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$ and since $\mathscr{A}_\sigma$ is closed under set differences (and S_1 contains $S_2, S_3, \ldots$) then $S_1 \setminus (S_1 \setminus \lim_{k\to\infty} S_k) = \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$. So $\mathscr{A}_\sigma$ is a monotone class, as claimed.

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Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

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So $S_1 \setminus \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$ and since $\mathscr{A}_\sigma$ is closed under set differences (and S_1 contains $S_2, S_3, \ldots$) then $S_1 \setminus (S_1 \setminus \lim_{k\to\infty} S_k) = \lim_{k\to\infty} S_k \in \mathscr{A}_\sigma$. So $\mathscr{A}_\sigma$ is a monotone class, as claimed.

Theorem II.1.6 (continued)

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof (continued). (b) If $\mathscr{A}$ is a Boolean algebra and a monotone class and $S_1, S_2, \ldots$ is an infinite sequence of sets from $\mathscr{A}$, then with $R_n = \sup_{k=1}^n S_k$, the sequence $R_1, R_2, \ldots$ is a monotonically increasing sequence in $\mathbb{A}$. Since $\mathscr{A}$ is a monotone class, then $\cup_{k=1}^{\infty} S_k = \cup_{k=1}^{\infty} R_k = \lim_{n\to\infty} R_n \in \mathscr{A}$. So $\mathscr{A}$ is a Boolean algebra closed under countable unions and hence $\mathscr{A}$ is a Boolean σ algebra, as claimed. $\square$

Theorem II.1.6 (continued)

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Proof (continued). (b) If $\mathscr{A}$ is a Boolean algebra and a monotone class and $S_1, S_2, \ldots$ is an infinite sequence of sets from $\mathscr{A}$, then with $R_n = \sup_{k=1}^{n} S_k$, the sequence $R_1, R_2, \ldots$ is a monotonically increasing sequence in $\mathbb{A}$. Since $\mathscr{A}$ is a monotone class, then $\cup_{k=1}^{\infty} S_k = \cup_{k=1}^{\infty} R_k = \lim_{n\to\infty} R_n \in \mathscr{A}$. So $\mathscr{A}$ is a Boolean algebra closed under countable unions and hence $\mathscr{A}$ is a Boolean σ algebra, as claimed. □

Theorem II.1.7

Theorem II.1.7. If $\mathscr{A}$ is a Boolean algebra and $\mathfrak{M}(\mathscr{A})$ is the monotone class generated by $\mathscr{A}$, then $\mathfrak{M}(\mathscr{A})$ is identical with the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by the family $\mathscr{A}$ of sets.

Proof. By Theorem II.1.6, $\mathscr{A}_\sigma(\mathscr{A})$ is a monotone class and by definition $\mathfrak{M}(\mathscr{A})$ is the smallest monotone class containing $\mathscr{A}$, so $\mathfrak{M}(\mathscr{A}) \subset \mathscr{A}_\sigma(\mathscr{A})$. We will show that $\mathfrak{M}(\mathscr{A})$ is a Boolean σ algebra containing $\mathscr{A}$. Since $\mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, then this will imply $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$ and hence $\mathscr{A}_\sigma(\mathscr{A}) = \mathfrak{M}(\mathscr{A})$.

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For $R \in \mathfrak{M}(\mathscr{A})$, denote by $\mathfrak{N}(R)$ the family of sets $S \in \mathfrak{M}(\mathscr{A})$ such that $S', R \cup S \in \mathfrak{M}(\mathscr{A})$. So by definition, $\mathfrak{N}(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$. If $S_1, S_2, \ldots \in \mathfrak{N}(R)$ is a monotone sequence then, since $\mathfrak{M}(\mathscr{A})$ is a monotone class, by Exercise II.1.6 $(\lim_{n\to\infty} S_n)^n = \lim_{n\to\infty} S_n' \in \mathfrak{M}(\mathscr{A})$, and the sequence of sets $R \cup S_1, R \cup S_2, \ldots \in \mathfrak{M}(\mathscr{A})$ is also a monotone sequence and, again by Exercise II.1.6, $(\lim_{n\to\infty} S_n) \cup R = \lim_{n\to\infty}(S_n \cup R) \in \mathfrak{M}(\mathscr{A})$. So, by the definition of $\mathfrak{N}(R)$, $\lim_{n\to\infty} S_n \in \mathfrak{N}(R)$ and so $\mathfrak{N}(R)$ is a monotone class.

Theorem II.1.7

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Proof. By Theorem II.1.6, $\mathscr{A}_\sigma(\mathscr{A})$ is a monotone class and by definition $\mathfrak{M}(\mathscr{A})$ is the smallest monotone class containing $\mathscr{A}$, so $\mathfrak{M}(\mathscr{A}) \subset \mathscr{A}_\sigma(\mathscr{A})$. We will show that $\mathfrak{M}(\mathscr{A})$ is a Boolean σ algebra containing $\mathscr{A}$. Since $\mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, then this will imply $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$ and hence $\mathscr{A}_\sigma(\mathscr{A}) = \mathfrak{M}(\mathscr{A})$.

For $R \in \mathfrak{M}(\mathscr{A})$, denote by $\mathfrak{N}(R)$ the family of sets $S \in \mathfrak{M}(\mathscr{A})$ such that $S', R \cup S \in \mathfrak{M}(\mathscr{A})$. So by definition, $\mathfrak{N}(\mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$. If $S_1, S_2, \ldots \in \mathfrak{N}(R)$ is a monotone sequence then, since $\mathfrak{M}(\mathscr{A})$ is a monotone class, by Exercise II.1.6 $(\lim_{n\to\infty} S_n)^n = \lim_{n\to\infty} S_n' \in \mathfrak{M}(\mathscr{A})$, and the sequence of sets $R \cup S_1, R \cup S_2, \ldots \in \mathfrak{M}(\mathscr{A})$ is also a monotone sequence and, again by Exercise II.1.6, $(\lim_{n\to\infty} S_n) \cup R = \lim_{n\to\infty}(S_n \cup R) \in \mathfrak{M}(\mathscr{A})$. So, by the definition of $\mathfrak{N}(R)$, $\lim_{n\to\infty} S_n \in \mathfrak{N}(R)$ and so $\mathfrak{N}(R)$ is a monotone class.

Theorem II.1.7 (continued 1)

Proof (continued). In addition, if $R \in \mathscr{A}$ then $\mathfrak{N}(R)$ is not empty because it contains $\mathscr{A}$, since $\mathscr{A}$ is a Boolean algebra. In this case, $\mathfrak{N}(R)$ is a monotone class containing $\mathscr{A}$ ad hence $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(R)$. Therefore $\mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$ for $R \in \mathscr{A}$. Since $\mathscr{A}$ is a Boolean algebra of sets then $\mathscr{X} = \mathfrak{M}(\mathscr{A})$. By the definition of $\mathfrak{N}(\mathscr{X})$, for every $S \in \mathfrak{N}(\mathscr{X}) = \mathfrak{M}(\mathscr{A})$ we have $S' \in \mathfrak{M}(\mathscr{A})$ so that $\mathfrak{M}(\mathscr{A})$ is closed under complements.

Furthermore, if $R \in \mathscr{A}$, then for an arbitrary $S \in \mathfrak{M}(\mathscr{A})$ we have $S \in \mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$, i.e., $R \cup S \in \mathfrak{M}(\mathscr{A})$; consequently (by the definition of $\mathfrak{N}(S)$), $R \in \mathfrak{N}(S)$. Therefore $\mathscr{A} \subset \mathfrak{N}(S)$. As shown above, $\mathfrak{N}(S)$ is a monotone class and, since it contains $\mathscr{A}$, we have $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(S)$. Since $\mathfrak{N}(S) \subset \mathfrak{M}(\mathscr{A})$ by definition, then $\mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ where S is any set in $\mathfrak{M}(\mathscr{A})$. So for any $R, S \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ we have (by the definition of $\mathfrak{N}(S)$) that $R \cup S, R' \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$. Therefore, $\mathfrak{M}(\mathscr{A})$ is a Boolean algebra.

Theorem II.1.7 (continued 1)

Proof (continued). In addition, if $R \in \mathscr{A}$ then $\mathfrak{N}(R)$ is not empty because it contains $\mathscr{A}$, since $\mathscr{A}$ is a Boolean algebra. In this case, $\mathfrak{N}(R)$ is a monotone class containing $\mathscr{A}$ ad hence $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(R)$. Therefore $\mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$ for $R \in \mathscr{A}$. Since $\mathscr{A}$ is a Boolean algebra of sets then $\mathscr{X} = \mathfrak{M}(\mathscr{A})$. By the definition of $\mathfrak{N}(\mathscr{X})$, for every $S \in \mathfrak{N}(\mathscr{X}) = \mathfrak{M}(\mathscr{A})$ we have $S' \in \mathfrak{M}(\mathscr{A})$ so that $\mathfrak{M}(\mathscr{A})$ is closed under complements.

Furthermore, if $R \in \mathscr{A}$, then for an arbitrary $S \in \mathfrak{M}(\mathscr{A})$ we have $S \in \mathfrak{N}(R) = \mathfrak{M}(\mathscr{A})$, i.e., $R \cup S \in \mathfrak{M}(\mathscr{A})$; consequently (by the definition of $\mathfrak{N}(S)$), $R \in \mathfrak{N}(S)$. Therefore $\mathscr{A} \subset \mathfrak{N}(S)$. As shown above, $\mathfrak{N}(S)$ is a monotone class and, since it contains $\mathscr{A}$, we have $\mathfrak{M}(\mathscr{A}) \subset \mathfrak{N}(S)$. Since $\mathfrak{N}(S) \subset \mathfrak{M}(\mathscr{A})$ by definition, then $\mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ where S is any set in $\mathfrak{M}(\mathscr{A})$. So for any $R, S \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$ we have (by the definition of $\mathfrak{N}(S)$) that $R \cup S, R' \in \mathfrak{N}(S) = \mathfrak{M}(\mathscr{A})$. Therefore, $\mathfrak{M}(\mathscr{A})$ is a Boolean algebra.

Theorem II.1.7 (continued 2)

Theorem II.1.7. If $\mathscr{A}$ is a Boolean algebra and $\mathfrak{M}(\mathscr{A})$ is the monotone class generated by $\mathscr{A}$, then $\mathfrak{M}(\mathscr{A})$ is identical with the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by the family $\mathscr{A}$ of sets.

Proof (continued). Since $\mathfrak{M}(\mathscr{A})$ is a Boolean algebra which is a monotone class then, by Theorem II.1.6, $\mathfrak{M}(\mathscr{A})$ is a Boolean σ algebra. The result now follows, as explained above. □