Theorem II.2.1

**Theorem II.2.1.** Every measure is continuous from above and below.

**Proof.** Suppose $R_1, R_2, \ldots$ is a monotonically increasing sequence from measure space $(\mathcal{X}, \mathcal{A}, \mu)$ and $\lim_{k \to \infty} R_k \in \mathcal{A}$. Then by defining $R_0 = \emptyset$ we have

$$\lim_{k \to \infty} R_k = \bigcup_{k=1}^{\infty} R_k = \bigcup_{k=1}^{\infty} (R_k \setminus R_{k-1}),$$

$R_k \setminus R_{k-1} \in \mathcal{A}$ for $k = 1, 2, \ldots$ by Theorem II.1.1(c), and the sets $R_k \setminus R_{k-1}$ are pairwise disjoint. So

$$\mu \left( \lim_{n \to \infty} R_k \right) = \mu \left( \bigcup_{k=1}^{\infty} (R_k \setminus R_{k-1}) \right)$$

$$= \sum_{k=1}^{\infty} \mu(R_k \setminus R_{k-1}) \text{ since a measure is countably additive}$$

$$= \lim_{k \to \infty} \left( \sum_{n=1}^{k} \mu(R_n \setminus R_{n-1}) \right)$$

**Theorem II.2.1 (continued 1)**

Proof (continued).

$$= \lim_{k \to \infty} \left( \sum_{n=1}^{k} \mu(R_n \setminus R_{n-1}) \right)$$

$$= \lim_{k \to \infty} \mu \left( \bigcup_{n=1}^{k} (R_n \setminus T_{n-1}) \right) \text{ since a measure is finite additive}$$

$$= \lim_{k \to \infty} \mu(R_k) \text{ since } \bigcup_{n=1}^{k} (R_n \setminus T_{n-1}) = R_k$$

because $R_1, R_2, \ldots$ is increasing.

So $\mu$ is continuous from below.

**Theorem II.2.1 (continued 2)**

Proof (continued).

$$\lim_{k \to \infty} (S_n \setminus S_k) = \bigcup_{k=1}^{\infty} (S_n \setminus S_k) = \bigcup_{k=1}^{\infty} (S_n \cap S_k^{c})$$

$$= S_n \cap \bigcup_{k=1}^{\infty} S_k^{c} \text{ by Lemma II.1.1}$$

$$= S_n \cap (\bigcap_{k=1}^{\infty} S_k)^{c} \text{ by Lemma II.1.2 (De Morgan)}$$

$$= S_n \cap \left( \lim_{k \to \infty} S_k^{c} \right) = S_n \setminus \lim_{k \to \infty} S_k$$

and by the first part of the proof (continuity from below)

$$\mu \left( \lim_{k \to \infty} (S_n \setminus S_k) \right) = \lim_{k \to \infty} \mu(S_n \setminus S_k).$$

Since

$$\mu(S_n \setminus S_k) = \mu(S_n) - \mu(S_k) \text{ for } k \geq n_0 \text{ by the “Excision Principle” (Exercise II.2.1(ii); this is where } \mu(S_n) < \infty \text{ is needed) we have}$$

$$\mu \left( S_n \setminus \lim_{k \to \infty} S_k \right) = \mu(S_n) - \mu \left( \lim_{k \to \infty} S_k \right) = \mu(S_n) - \lim_{k \to \infty} \mu(S_k). \quad (*)$$
Theorem II.2.1. Every measure is continuous from above and below.

Proof (continued). Therefore
\[
\mu \left( \lim_{k \to \infty} S_k \right) = \mu \left( S_{R_0} \setminus \left( S_{R_0} \setminus \lim_{k \to \infty} S_k \right) \right)
\]
since the sequence is decreasing
\[
= \mu(S_{R_0}) - \mu \left( S_{R_0} \setminus \lim_{k \to \infty} S_k \right)
\]
by the Excision Principle (Exercise II.2.1(ii))
\[
= \lim_{k \to \infty} \mu(S_k) \text{ by (*)).}
\]
\[
\square
\]

Theorem II.2.2. Every finite, nonnegative, additive set function \( F \) on a Boolean \( \sigma \) algebra \( \mathcal{A} \) and satisfying \( F(\emptyset) = 0 \), which is either continuous from below at every \( R \in \mathcal{A} \) or continuous from above at \( \emptyset \in \mathcal{A} \), is necessarily also \( \sigma \) additive or "countably additive" (i.e., \( \mu \) is a measure).

Proof. Let \( S_1, S_2, \ldots \) be any infinite sequence of disjoint sets from \( \mathcal{A} \). Then the sequence \( R_1, R_2, \ldots \) with \( R_n = \bigcup_{k=1}^{n} S_k \) is monotonically increasing. Since \( F \) is additive by hypothesis then
\[
F(R_n) = R \left( \bigcup_{k=1}^{n} S_k \right) = \sum_{k=1}^{n} F(S_k).
\]
If \( F \) is continuous from below at \( R \in \mathcal{A} \) then
\[
F \left( \bigcup_{k=1}^{\infty} S_k \right) = F \left( \lim_{n \to \infty} R_n \right) = \lim_{n \to \infty} F(R_n) = \lim_{n \to \infty} \left( \sum_{k=1}^{n} F(S_k) \right) = \sum_{k=1}^{\infty} F(S_k).
\]
So \( F \) is countably additive.

Lemma II.2.A. Let \( \mathcal{A} \) be a Boolean algebra on set \( \mathcal{X} \) and let \( \mu \) be a measure on \( \mathcal{A} \). Define extended real-valued set function
\[
\mu^+(R) = \inf \left\{ \sum_{k=1}^{\infty} \mu(S_k) \mid R \subset \bigcup_{k=1}^{\infty} S_k, S_k \in \mathcal{A} \text{ for all } k \in \mathbb{N} \right\}
\]
on the power set \( \mathcal{P}_{\mathcal{A}} \) of \( \mathcal{A} \). Sets \( S_1, S_2, \ldots \in \mathcal{A} \) such that \( R \subset \bigcup_{k=1}^{\infty} S_k \) are said to cover \( R \). For any \( R_1, R_2, \ldots \in \mathcal{P}_{\mathcal{A}} \) we have
\[
\mu^+ \left( \bigcup_{n=1}^{\infty} R_n \right) \leq \sum_{n=1}^{\infty} \mu^+(R_n).
\]
Proof. Let \( \varepsilon > 0 \). For each \( R_n \) there is a covering \( S_{n1}, S_{n2}, \ldots \in \mathcal{A} \) such that \( R_n \subset \bigcup_{k=1}^{\infty} S_{nk} \) and
\[
\mu^+(R_n) \leq \sum_{k=1}^{\infty} \mu(S_{nk}) \leq \mu^+(R_n) + \frac{\varepsilon}{2^n},
\]
by the infimum definition of \( \mu^+ \).
Lemma II.2.A (continued)

Proof (continued). Now \( \{S_{nk} \mid n, k \in \mathbb{N}\} \) is a countable family of sets in \( \mathcal{A} \) and \( \bigcup_{n=1}^{\infty} R_n \subset \bigcup_{n,k=1}^{\infty} S_{nk} \), so

\[
\mu^+ \left( \bigcup_{n=1}^{\infty} R_n \right) \leq \sum_{n,k=1}^{\infty} \mu(S_{nk}) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(S_{nk}) \leq \sum_{n=1}^{\infty} \left( \mu^+(R_n) + \frac{\varepsilon}{s^n} \right) = \varepsilon + \sum_{n=1}^{\infty} \mu^+(R_n).
\]

Since \( \varepsilon > 0 \) is arbitrary, the claim follows. \( \square \)

Lemma II.2.1

Lemma II.2.1. If \( M \) is an outer measure on the power set \( \mathcal{G}_X \) of \( X \), then the class \( \mathcal{A}_M \) of all \( M \)-measurable sets \( S \in \mathcal{G}_X \) is a Boolean \( \sigma \)-algebra, and the outer measure \( M \) restricted to \( \mathcal{A}_M \) is a measure.

Proof. First, we prove \( \mathcal{A}_M \) is a Boolean algebra. For any \( R \in \mathcal{G}_X \), we have

\[
M(R) = M(R \cap \mathcal{G}_X) = M(R \cap X) = M(R \cap \mathcal{G}_X) = M(R \cap X) + M(R \cap \mathcal{G}_X)
\]

and so \( \emptyset \in \mathcal{A}_M \). Clearly if \( S \) is measurable and satisfies the Carathéodory condition, then \( S' \) satisfies the Carathéodory condition and is measurable, that is, if \( S \in \mathcal{A}_M \) then \( S' \in \mathcal{A}_M \). Now let \( S_1, S_2 \in \mathcal{A}_M \). Then for any \( R \in \mathcal{G}_X \) we have

\[
M(R) = M(R \cap S_1) + M(R \cap S_2'). \quad (2.11)
\]

With \( R \cap S_1, R \cap S_2' \in \mathcal{G}_X \) and the fact that \( S_2 \) is measurable then we have

\[
M(R \cap S_1) = M((R \cap S_1) \cap S_2) + M((R \cap S_1) \cap S_2')
\]

and

\[
M(R \cap S_2') = M((R \cap S_2') \cap S_1) + M((R \cap S_2') \cap S_2).
\]

Lemma II.2.1 (continued 1)

Proof (continued). Substituting these into (2.11) we get

\[
M(R) = M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) + M(R \cap S_1' \cap S_2')
\]

Now this equation holds for all \( R \in \mathcal{G}_X \). So by replacing \( R \) by

\[
M(R \cap (S_1 \cap S_2)) = M((R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2) + M((R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2')
\]

and

\[
M(R \cap (S_1 \cap S_2')) = M((R \cap (S_1 \cup S_2')) \cap S_1 \cap S_2) + M((R \cap (S_1 \cup S_2')) \cap S_1 \cap S_2')
\]

for all \( R \in \mathcal{G}_X \), so that \( S_1 \cup S_2 \in \mathcal{A}_M \). Therefore \( \mathcal{A}_M \) is a Boolean algebra.

Lemma II.2.1 (continued 2)

Proof (continued). . . by Lemma II.1.1,

\[
(R \cap S_1 \cup S_2') \cap S_1 \cap S_2 = R \cap S_1 \cap S_2 \text{ since } (S_1 \cup S_2') \cap S_1 \cap S_2 = S_1 \cap S_2,
\]

\[
(R \cap S_1 \cup S_2') \cap S_1' \cap S_2 = R \cap S_1' \cap S_2' \text{ since } (S_1 \cup S_2') \cap S_1 \cap S_2' = S_1 \cap S_2',
\]

\[
(R \cap S_1 \cup S_2') \cap S_1 \cap S_2 = R \cap S_1' \cap S_2 \text{ since } (S_1 \cup S_2') \cap S_1 \cap S_2' = S_1 \cap S_2,
\]

and

\[
(R \cap S_1 \cup S_2') \cap (S_1 \cap S_2)' = \emptyset \text{ since } (S_1 \cup S_2') \cap (S_1 \cap S_2)' = \emptyset.
\]

Replacing part of the right hand side of (2.14) with \( M(R \cap (S_1 \cup S_2')) \) we get

\[
M(R) = M((R \cap (S_1 \cup S_2')) \cap S_1 \cap S_2) + M(R \cap (S_1 \cup S_2')'),
\]

since . . .
Lemma II.2.1 (continued 3)

**Proof (continued).** To prove that \( \mathcal{A}_M \) is a Boolean \( \sigma \) algebra, we consider a sequence of sets \( S_1, S_2, \ldots \in \mathcal{A}_M \). We can assume without loss of generality that the sets are disjoint (or we can use the Boolean algebra properties to create a disjoint sequence from a non-disjoint sequence). With \( S_1 \) and \( S_2 \) disjoint we have \( S_1 \cap S_2' = S_2 \) and \( S_1' \cap S_2 = S_2' \) so that (2.15) becomes

\[
M(R \cap (S_1 \cup S_2)) = M(R \cap S_1) + M(R \cap S_2)
\]

and inductively we have

\[
M(R \cap (\bigcup_{k=1}^{n} S_k)) = \sum_{k=1}^{n} M(R \cap S_k). \tag{2.17}
\]

Since \( M \) is an outer measure and so monotone (by subadditivity and nonnegativity) we have

\[
M\left(R \cap \left( \bigcup_{k=1}^{n} S_k \right) \right) \geq M\left(R \cap \left( \bigcup_{k=1}^{n} S_k \right) \right). \tag{2.18}
\]

Lemma II.2.1 (continued 4)

**Proof (continued).** Now \( \bigcup_{k=1}^{n} S_k \) is \( M \) measurable since \( \mathcal{A}_M \) is a Boolean algebra, so by the Carathéodory splitting condition

\[
M(R) = M(R \cap (\bigcup_{k=1}^{n} S_k)) + M(R \cap (\bigcup_{k=1}^{n} S_k)')
\]

\[
= M(\bigcup_{k=1}^{n} (R \cap S_k)) + M(R \cap (\bigcup_{k=1}^{n} S_k)') \tag{by Lemma II.1.1}
\]

\[
\geq \sum_{k=1}^{n} M(R \cap S_k) + M(R \cap (\bigcup_{k=1}^{n} S_k)') \tag{by (2.17) and (2.18)}.
\]

Since this holds for arbitrary \( n \in \mathbb{N} \) then

\[
M(R) \geq \sum_{k=1}^{\infty} M(R \cap S_k) + M(R \cap (\bigcup_{k=1}^{\infty} S_k)') \tag{(*)}
\]

As an outer measure \( M \) is countably subadditive,

\[
M(R \cap (\bigcup_{k=1}^{\infty} S_k)) = M(\bigcup_{k=1}^{\infty} (R \cap S_k)) \leq \sum_{k=1}^{\infty} M(R \cap S_k).
\]

Lemma II.2.2

**Lemma II.2.2.** Every set \( R \) in the Boolean \( \sigma \) algebra \( \mathcal{A}_\sigma = \mathcal{A}_\sigma(\mathcal{A}) \) generated by \( \mathcal{A} \) is \( \mu^+ \) measurable. That is, \( \mathcal{A}_\sigma \subset \mathcal{A}_\mu^+ \).

**Proof.** Let \( R \in \mathcal{A}_\mathcal{A}' \). By the infimum definition of \( \mu^+ \), for any \( \varepsilon > 0 \) there is sequence \( S_1, S_2, \ldots \in \mathcal{A} \) such that \( R \subset \bigcup_{k=1}^{\infty} S_k \) and

\[
\mu^+(R) \leq \sum_{k=1}^{\infty} \mu(S_k) \leq \mu^+(R) + \varepsilon \tag{2.23}
\]

(here, \( \mu \) is a measure on \( \mathcal{A} \) that defines outer measure \( \mu^+ \)). For any \( S \in \mathcal{A} \) we have (since \( S_k = (S_k \cap S) \cup (S_k \cap S') \) and \( \mu \) is a measure on \( \mathcal{A} \) and hence countable additivity),

\[
\sum_{k=1}^{\infty} \mu(S_k) = \sum_{k=1}^{\infty} \mu((S_k \cap S) \cup (S_k \cap S')) = \sum_{k=1}^{\infty} \mu(S_k \cap S) + \mu(S_k \cap S')
\]

\[
= \sum_{k=1}^{\infty} \mu(S_k \cap S) + \sum_{k=1}^{\infty} \mu(S_k \cap S') \geq \mu^+(R \cap S) \mu^+(R \cap S') \tag{2.24}
\]
Lemma II.2.2. Every set $R$ in the Boolean $\sigma$ algebra $\mathcal{A}_\sigma = \mathcal{A}_\sigma(\mathcal{A})$ generated by $\mathcal{A}$ is $\mu^+$ measurable. That is, $\mathcal{A}_\sigma \subset \mathcal{A}_{\mu^+}$.

Proof. ... since $R \cap S \subset \bigcup_{k=1}^{\infty} (S_k \cap S)$ and $R \cap S' \subset \bigcup_{k=1}^{\infty} (S_k \cap S')$. Combining (2.23) and (2.24) we obtain

$\mu^+(R) + \varepsilon \geq \mu^+(R \cap S) + \mu^+(R \cap S')$ or, since $\varepsilon > 0$ is arbitrary,

$\mu^+(R) \geq \mu^+(R \cap S) + \mu^+(R \cap S')$. By subadditivity, we can reverse this inequality and conclude that $S$ is measurable; i.e., $S \in \mathcal{A}_{\mu^+}$. Since $S$ is an arbitrary element of $\mathcal{A}$, then $\mathcal{A} \subset \mathcal{A}_{\mu^+}$.

So $\mathcal{A}_{\mu^+}$ is a Boolean $\sigma$ algebra by Lemma II.2.1, and $\mathcal{A}_\sigma = \mathcal{A}_\sigma(\mathcal{A})$ is the smallest $\sigma$ algebra containing $\mathcal{A}$, so we must have $\mathcal{A}_\sigma \subset \mathcal{A}_{\mu^+}$, as claimed. \qed