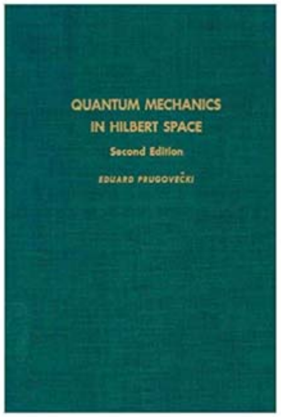
Modern Algebra

Chapter II. Measure Theory and Hilbert Spaces of Functions

II.2. Measures and Measure Spaces—Proofs of Theorems



Theorem II.2.1

Theorem II.2.1. Every measure is continuous from above and below.

Proof. Suppose $R_1, R_2, \ldots$ is a monotonically increasing sequence from measure space $(\mathscr{X}, \mathscr{A}, \mu)$ and $\lim_{k\to\infty} R_k \in \mathscr{A}$. Then by defining $R_0 = \varnothing$ we have

$$\lim_{k\to\infty} R_k = \cup_{k=1}^{\infty} R_k = \cup_{k=1}^{\infty}(R_k \setminus R_{k-1}),$$

$R_k \setminus R_{k-1} \in \mathscr{A}$ for $k = 1, 2, \ldots$ by Theorem II.1.1(c), and the sets $R_k \setminus R_{k-1}$ are pairwise disjoint. So

$$\begin{aligned} \mu\left(\lim_{n\to\infty} R_k\right) &= \mu\left(\cup_{k=1}^{\infty}(R_k \setminus R_{k-1})\right) \\ &= \sum_{k=1}^{\infty} \mu(R_k \setminus R_{k-1}) \text{ since a measure is countably additive} \\ &= \lim_{k\to\infty}\left(\sum_{n=1}^{k} \mu(R_n \setminus R_{n-1})\right) \end{aligned}$$

Theorem II.2.1 (continued 1)

Proof (continued).

$$\begin{aligned} &= \lim_{k\to\infty}\left(\sum_{n=1}^{k} \mu(R_n \setminus R_{n-1})\right) \\ &= \lim_{k\to\infty} \mu\left(\cup_{n=1}^{k}(R_n \setminus T_{n-1})\right) \text{ since a measure is finite additive} \\ &= \lim_{k\to\infty} \mu(R_k) \text{ since } \cup_{n=1}^{k}(R_n \setminus T_{n-1}) = R_k \\ &\quad \text{because } R_1, R_2, \ldots \text{ is increasing.} \end{aligned}$$

So μ is continuous from below.

Suppose $S_1, S_2, \ldots \in \mathscr{A}$ is a monotonically decreasing sequence and $\mu(Sn_0) < \infty$ for some n_0. Then $S_{n_0} \setminus S_1, S_{n_0} \setminus S_2, \ldots$ is monotonically increasing, so if $\lim_{k\to\infty} S_k \in \mathscr{A}$ then ...

Theorem II.2.1 (continued 2)

Proof (continued).

$$\begin{aligned} \lim_{k\to\infty}(S_{n_0} \setminus S_k) &= \cup_{k=1}^{\infty}(S_{n_0} \setminus S_k) = \cup_{k=1}^{\infty}(S_{n_0} \cap S_k^c) \\ &= S_{n_0} \cap (\cup_{k=1}^{\infty} S_k^c) \text{ by Lemma II.1.1} \\ &= S_{n_0} \cap (\cap_{k=1}^{\infty} S_k)^c \text{ by Lemma II.1.2 (De Morgan)} \\ &= S_{n_0} \cap \left(\lim_{k\to\infty} S_k\right)^c - S_{n_0} \setminus \lim_{k\to\infty} S_k \end{aligned}$$

and by the first part of the proof (continuity from below) $\mu\left(\lim_{k\to\infty}(S_{n_0} \setminus S_k)\right) = \lim_{k\to\infty} \mu(S_{n_0} \setminus S_k)$. Since $\mu(S_{n_0} \setminus S_k) = \mu(S_{n_0}) - \mu(S_k)$ for $k \geq n_0$ by the "Excision Principle" (Exercise II.2.1(ii); this is where $\mu(S_{n_0}) < \infty$ is needed) we have

$$\mu\left(S_{n_0} \setminus \lim_{k\to\infty} S_k\right) = \mu(S_{n_0}) - \mu\left(\lim_{k\to\infty} S_k\right) = \mu(S_{n_0}) = \lim_{k\to\infty} \mu(S_k). \quad (*)$$

Theorem II.2.1 (continued 3)

Theorem II.2.1. Every measure is continuous from above and below.

Proof (continued). Therefore

$$\begin{aligned}\mu\left(\lim_{k\to\infty} S_k\right) &= \mu\left(S_{n_0}\setminus\left(S_{n_0}\setminus \lim_{k\to\infty} S_k\right)\right)\\ &\quad\text{since the sequence is decreasing}\\ &= \mu(S_{n_0}) - \mu\left(S_{n_0}\setminus\lim_{k\to\infty} S_k\right)\\ &\quad\text{by the Excision Principle (Exercise II.2.1(ii))}\\ &= \lim_{k\to\infty}\mu(S_k) \text{ by } (*).\end{aligned}$$

□

Theorem II.2.2

Theorem II.2.2. Every finite, nonnegative, additive set function F on a Boolean σ algebra $\mathcal{A}$ and satisfying $F(\varnothing)=0$, which is either continuous from below at every $R\in\mathcal{A}$ or continuous from above at $\varnothing\in\mathcal{A}$, is necessarily also σ additive or "countably additive" (i.e., μ is a measure).

Proof. Let $S_1, S_2, \ldots$ be any infinite sequence of disjoint sets from $\mathscr{A}$. Then the sequence $R_1, R_2, \ldots$ with $R_n = \cup_{k=1}^n S_k$ is monotonically increasing. Since F is additive by hypothesis then

$$F(R_n) = R\left(\cup_{k=1}^n S_k\right) = \sum_{k=1}^n F(S_k).$$

If F is continuous from below at $R\in\mathscr{A}$ then

$$F\left(\cup_{k=1}^\infty S_k\right) = F\left(\lim_{n\to\infty} R_n\right) = \lim_{n\to\infty} F(R_n) - \lim_{n\to\infty}\left(\sum_{k=1}^n F(S_k)\right) = \sum_{k=1}^\infty F(S_k).$$

So F is countably additive.

Theorem II.2.2 (continued)

Theorem II.2.2. Every finite, nonnegative, additive set function F on a Boolean σ algebra $\mathcal{A}$ and satisfying $F(\varnothing)=0$, which is either continuous from below at every $R\in\mathcal{A}$ or continuous from above at $\varnothing\in\mathcal{A}$, is necessarily also σ additive or "countably additive" (i.e., μ is a measure).

Proof (continued). If F is continuous form above at $\varnothing$ then, since $R\setminus R_1, R\setminus R_2, \ldots$ is increasing with

$$\lim_{n\to\infty}(R\setminus R_n) = \lim_{n\to\infty}\left(\cup_{k=1}^\infty S_k\setminus\cup_{k=1}^n S_k\right) = \lim_{n\to\infty}\left(\cup_{k=n+1}^\infty S_k\right) = \varnothing,$$

then we have

$$0 = F(\varnothing) = F\left(\lim_{n\to\infty}(R\setminus R_n)\right) = \lim_{n\to\infty} F(R\setminus R_n) = F(R) = \lim_{n\to\infty} F(R_n).$$

That is, $F\left(\cup_{k=1}^\infty S_k\right) = F(R) = \lim_{n\to\infty} F(R_n) = \sum_{k=1}^\infty F(S_k)$. □

Lemma II.2.A

Lemma II.2.A. Let $\mathscr{A}$ be a Boolean algebra on set $\mathscr{X}$ and let μ be a measure on $\mathscr{A}$. Define extended real-valued set function

$$\mu^+(R) = \inf\left\{\sum_{k=1}^\infty \mu(S_k) \;\middle|\; R\subset\cup_{k=1}^\infty S_k, S_k\in\mathscr{A} \text{ for all } k\in\mathbb{N}\right\}$$

on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$. Sets $S_1, S_2, \ldots \in\mathscr{A}$ such that $R\subset\cup_{k=1}^\infty S_k$ are said to *cover* R. For any $R_1, R_2, \ldots\in\mathscr{G}_{\mathscr{X}}$ we have

$$\mu^+\left(\cup_{n=1}^\infty R_n\right) \le \sum_{n=1}^\infty \mu^+(R_n).$$

Proof. Let $\varepsilon>0$. For each R_n there is a covering $S_{n1}, S_{n2}, \ldots\in\mathscr{A}$ such that $R_n\subset\cup_{k=1}^\infty S_{nk}$ and

$$\mu^+(R_n) \le \sum_{k=1}^\infty \mu(S_{nk}) \le \mu^+(R_n) + \frac{\varepsilon}{2^n},$$

by the infimum definition of μ^+.

Lemma II.2.A (continued)

Proof (continued). Now $\{S_{nk} \mid n,k \in \mathbb{N}\}$ is a countable family of sets in $\mathscr{A}$ and $\cup_{n=1}^{\infty} R_n \subset \cup_{n,k=1}^{\infty} S_{nk}$, so

$$\mu^+\left(\cup_{n=1}^{\infty} R_n\right) \le \sum_{n,k=1}^{\infty} \mu(S_{nk}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(S_{nk})$$

$$\le \sum_{k=1}^{\infty}\left(\mu^+(R_n) + \frac{\varepsilon}{s^n}\right) = \varepsilon + \sum_{n=1}^{\infty} \mu^+(R_n),$$

Since $\varepsilon > 0$ is arbitrary, the claim follows. □

Lemma II.2.1

Lemma II.2.1. If M is an outer measure on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$ then the class $\mathscr{A}_M$ of all M-measurable sets $S \in \mathscr{G}_{\mathscr{X}}$ is a Boolean σ algebra, and the outer measure M restricted to $\mathscr{A}_M$ is a measure.

Proof. First, we prove $\mathscr{A}_M$ is a Boolean algebra. For any $R \in \mathscr{G}_{\mathscr{X}}$ we have

$$M(R) = M(R \cap \mathscr{X}) = 0 + M(R \cap \mathscr{X}) = M(R \cap \varnothing) + M(R \cap \mathscr{X})$$

and so $\varnothing \in \mathscr{A}_M$. Clearly if S is measurable and satisfies the Carathéodory condition, then S' satisfies the Carathéodory condition and is measurable; that is, if $S \in \mathscr{A}_M$ then $S' \in \mathscr{A}_M$. Now let $S_1, S_2 \in \mathscr{A}_M$. Then for any $R \in \mathscr{G}_{\mathscr{X}}$ we have

$$M(R) = M(R \cap S_1) + M(R \cap S_1'). \qquad (2.11)$$

With $R \cap S_1, R \cap S_1' \in \mathscr{G}_{\mathscr{X}}$ and the fact that S_2 is measurable then we have

$$M(R \cap S_1) = M((R \cap S_1) \cap S_2) + M((R \cap S_1) \cap S_2')$$

$$\text{and } M(R \cap S_1') = M((R \cap S_1') \cap S_2) + M((R \cap S_1') \cap S_2').$$

Lemma II.2.1 (continued 1)

Proof (continued). Substituting these into (2.11) we get

$$\begin{aligned} M(R) &= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \\ &\quad + M(R \cap S_1' \cap S_2') \\ &= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \\ &\quad + M(R \cap (S_1 \cup S_2)') \text{ by Lemma II.1.2 (De Morgan)} \qquad (2.14) \end{aligned}$$

Now this equation holds for all $R \in \mathscr{G}_{\mathscr{X}}$. So by replacing R by $R \cap (S_1 \cap S_2)$ we get

$$M(R \cap (S_1 \cap S_2)) = M((R \cap (S_1 \cup S_2) \cap S_1 \cap S_2) + M((R \cap (S_1 \cup S_2) \cap S_1 \cap S_2')$$

$$+ M((R \cap (S_1 \cup S_2)) \cap S_1' \cap S_2) + M((R \cap (S_1 \cup S_2)) \cap (S_1 \cup S_2)')$$

$$= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \qquad (2.15)$$

since...

Lemma II.2.1 (continued 2)

Proof (continued). ...by Lemma II.1.1,

$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2 = R \cap S_1 \cap S_2$ since $(S_1 \cup S_2) \cap S_1 \cap S_2 = S_1 \cap S_2$,

$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2' = R \cap S_1 \cap S_2'$ since $(S_1 \cup S_2) \cap S_1 \cap S_2' = S_1 \cap S_2'$,

$(R \cap (S_1 \cup S_2)) \cap S_1' \cap S_2 = R \cap S_1' \cap S_2$ since $(S_1 \cup S_2) \cap S_1' \cap S_2 = S_1' \cap S_2$,

and $(R \cap (S_1 \cup S_2)) \cap (S_1 \cup S_2)' = \varnothing$ since $(S_1 \cup S_2) \cap (S_1 \cup S_2)' = \varnothing$.

Replacing part of the right hand side of (2.14) with $M(R \cap (S_1 \cup S_2)')$ we get

$$M(R) = M(R \cap (S_1 \cap S_2)) + M(R \cap (S_1 \cup S_2)')$$

for all $R \in \mathscr{G}_{\mathscr{X}}$, so that $S_1 \cup S_2 \in \mathscr{A}_M$. Therefore $\mathscr{A}_M$ is a Boolean algebra.

Lemma II.2.1 (continued 3)

Proof (continued). To prove the $\mathscr{A}_M$ is a Boolean σ algebra, we consider a sequence of sets $S_1, S_2, \ldots \in \mathscr{A}_M$. We can assume without loss of generality that the sets are disjoint (or we can use the Boolean algebra properties to create a disjoint sequence from a non-disjoint sequence). With S_1 and S_2 disjoint we have $S_1 \cap S_2' = S_2$ and $S_1' \cap S_2 = S_2$ so that (2.15) becomes

$$M(R \cap (S_1 \cup S_2)) = M(R \cap S_1) + M(R \cap S_2)$$

and inductively we have

$$M\left(R \cap (\cup_{n=1}^{\infty} S_k)\right) = \sum_{k=1}^{n} M(R \cap S_k). \qquad (2.17)$$

Since M is an outer measure and so monotone (by subadditivity and nonnegativity) we have

$$M\left(R \cap (\cup_{k=1}^{n} S_k)'\right) \geq M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right). \qquad (2.18)$$

Lemma II.2.1 (continued 4)

Proof (continued). Now $\cup_{k=1}^{n} S_k$ is M measurable since $\mathscr{A}_M$ is a Boolean algebra, so by the Carathéodory splitting condition

$$\begin{aligned} M(R) &= M\left(R \cap (\cup_{k=1}^{n} S_k)\right) + M\left(R \cap (\cup_{k=1}^{n} S_k)'\right) \\ &= M\left(\cup_{k=1}^{n} (R \cap S_k)\right) + M\left(R \cap (\cup_{k=1}^{n} S_k)'\right) \text{ by Lemma II.1.1} \\ &\geq \sum_{k=1}^{n} M(R \cap S_k) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right) \text{ by (2.17) and (2.18).} \end{aligned}$$

Since this holds for arbitrary $n \in \mathbb{N}$ then

$$M(R) \geq \sum_{k=1}^{\infty} M(R \cap S_k) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right). \qquad (*)$$

As an outer measure M is countably subadditive,

$$M\left(R \cap (\cup_{k=1}^{\infty} S_k)\right) = M\left(\cup_{k=1}^{\infty} (R \cap S_k)\right) \leq \sum_{k=1}^{\infty} M(R \cap S_k).$$

Lemma II.2.1 (continued 5)

Proof (continued). So from this and $(*)$ we have

$$\begin{aligned} M(R) &\geq \sum_{k=1}^{\infty} M(R \cap S_k) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right) \\ &\geq M\left(R \cap (\cup_{k=1}^{\infty} S_k)\right) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right). \end{aligned}$$

By subadditivity,

$$\begin{aligned} M(R) &= M\left((R \cap (\cup_{k=1}^{\infty} S_k)) \cup (R \cap (\cup_{k=1}^{\infty} S_k)')\right) \\ &\leq M\left(R \cap (\cup_{k=1}^{\infty} S_k)\right) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right), \end{aligned}$$

hence $M(R) = M\left(R \cap (\cup_{k=1}^{\infty} S_k)\right) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right)$ for all $R \in \mathscr{G}_{\mathscr{X}}$ so that $\cup_{k=1}^{\infty} S_k$ is M measurable. That is, $\cup_{k=1}^{\infty} S_k \in \mathscr{A}_M$ and so $\mathscr{A}_M$ is a Boolean σ algebra.
Finally, to prove that M is σ additive on $\mathscr{A}_M$ (and hence is a measure on $\mathscr{A}_M$), take $R = \cup_{k=1}^{\infty} S_k$ is (2,21) to get $M\left(\cup_{k=1}^{\infty} S_n\right) \geq \sum_{k=1}^{\infty} \mu(S_k)$. Countable additivity shows the reverse of this inequality and hence M is σ additive. □

Lemma II.2.2

Lemma II.2.2. Every set R in the Boolean σ algebra $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is μ^+ measurable. That is, $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$.

Proof. Let $R \in \mathscr{G}_{\mathscr{X}}$. By the infimum definition of μ^+, for any $\varepsilon > 0$ there is sequence $S_1, S_2, \ldots \in \mathscr{A}$ such that $R \subset \cup_{k=1}^{\infty} S_k$ and

$$\mu^+(R) \leq \sum_{k=1}^{\infty} \mu(S_k) \leq \mu^+(R) + \varepsilon \qquad (2.23)$$

(here, μ is a measure on $\mathscr{A}$ that defines outer measure μ^+). For any $S \in \mathscr{A}$ we have (since $S_k = (S_k \cap S) \uplus (S_k \cap S')$ and μ is a measure on $\mathscr{A}$ and hence countable additivity),

$$\sum_{k=1}^{\infty} \mu(S_k) = \sum_{k=1}^{\infty} \mu((S_k \cap S) \uplus (S_k \cap S')) = \sum_{k=1}^{\infty} (\mu(S_k \cap S) + \mu(S_k \cap S'))$$

$$= \sum_{k=1}^{\infty} \mu(S_k \cap S) + \sum_{k=1}^{\infty} \mu(S_k \cap S') \geq \mu^+(R \cap S) + \mu^+(R \cap S') \qquad (2.24)$$

Lemma II.2.2 (continued)

Lemma II.2.2. Every set R in the Boolean σ algebra $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is μ^+ measurable. That is, $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$.

Proof. ...since $R \cap S \subset \cup_{k=1}^\infty (S_n \cap S)$ and $R \cap S' \subset \cup_{k=1}^\infty (S_k \cap S')$. Combining (2.23) and (2.24) we obtain $\mu^+(R) + \varepsilon \geq \mu^+(R \cap S) + \mu^+(R \cap S')$ or, since $\varepsilon > 0$ is arbitrary, $\mu^+(R) \geq \mu^+(R \cap S) + \mu^+(R \cap S')$. By subadditivity, we can reverse this inequality and conclude that S is measurable; i.e., $S \in \mathscr{A}_{\mu^+}$. Since S is an arbitrary element of $\mathscr{A}$, then $\mathscr{A} \subset \mathscr{A}_{\mu^+}$.

So $\mathscr{A}_{\mu^+}$ is a Boolean σ algebra by Lemma II.2.1, and $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, so we must have $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$, as claimed. □