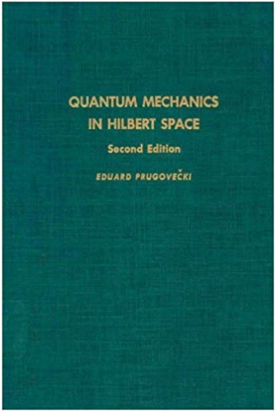
Modern Algebra

Chapter II. Measure Theory and Hilbert Spaces of Functions

II.2. Measures and Measure Spaces—Proofs of Theorems



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Theorem II.2.1

Theorem II.2.1. Every measure is continuous from above and below.

Proof. Suppose $R_1, R_2, \ldots$ is a monotonically increasing sequence from measure space $(\mathscr{X}, \mathscr{A}, \mu)$ and $\lim_{k\to\infty} R_k \in \mathscr{A}$. Then by defining $R_0 = \varnothing$ we have

$$\lim_{k\to\infty} R_k = \cup_{k=1}^{\infty} R_k = \dot\cup_{k=1}^{\infty}(R_k \setminus R_{k-1}),$$

$R_k \setminus R_{k-1} \in \mathscr{A}$ for $k = 1, 2, \ldots$ by Theorem II.1.1(c), and the sets $R_k \setminus R_{k-1}$ are pairwise disjoint.

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$$\begin{aligned} \mu\left(\lim_{n\to\infty} R_k\right) &= \mu\left(\cup_{k=1}^{\infty}(R_k \setminus R_{k-1})\right) \\ &= \sum_{k=1}^{\infty} \mu(R_k \setminus R_{k-1}) \text{ since a measure is countably additive} \\ &= \lim_{k\to\infty} \left(\sum_{n=1}^{k} \mu(R_n \setminus R_{n-1})\right) \end{aligned}$$

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Theorem II.2.1 (continued 1)

Proof (continued).

$$= \lim_{k\to\infty}\left(\sum_{n=1}^{k}\mu(R_n \setminus R_{n-1})\right)$$

$$= \lim_{k\to\infty}\mu\left(\cup_{n=1}^{k}(R_n \setminus T_{n-1})\right) \text{ since a measure is finite additive}$$

$$= \lim_{k\to\infty}\mu(R_k) \text{ since } \cup_{n=1}^{k}(R_n \setminus T_{n-1}) = R_k$$

because $R_1, R_2, \ldots$ is increasing.

So μ is continuous from below.

Suppose $S_1, S_2, \ldots \in \mathscr{A}$ is a monotonically decreasing sequence and $\mu(Sn_0) < \infty$ for some n_0. Then $S_{n_0} \setminus S_1, S_{n_0} \setminus S_2, \ldots$ is monotonically increasing, so if $\lim_{k\to\infty} S_k \in \mathscr{A}$ then ...

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Proof (continued).

$$\begin{aligned} &= \lim_{k\to\infty}\left(\sum_{n=1}^{k}\mu(R_n\setminus R_{n-1})\right) \\ &= \lim_{k\to\infty}\mu\left(\cup_{n=1}^{k}(R_n\setminus T_{n-1})\right) \text{ since a measure is finite additive} \\ &= \lim_{k\to\infty}\mu(R_k) \text{ since } \cup_{n=1}^{k}(R_n\setminus T_{n-1}) = R_k \\ &\quad \text{because } R_1, R_2, \ldots \text{ is increasing.} \end{aligned}$$

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Suppose $S_1, S_2, \ldots \in \mathscr{A}$ is a monotonically decreasing sequence and $\mu(Sn_0) < \infty$ for some n_0. Then $S_{n_0}\setminus S_1, S_{n_0}\setminus S_2, \ldots$ is monotonically increasing, so if $\lim_{k\to\infty} S_k \in \mathscr{A}$ then ...

Theorem II.2.1 (continued 2)

Proof (continued).

$$\begin{aligned}
\lim_{k\to\infty}(S_{n_0}\setminus S_k) &= \cup_{k=1}^{\infty}(S_{n_0}\setminus S_k) = \cup_{k=1}^{\infty}(S_{n_0}\cap S_k^c) \\
&= S_{n_0}\cap(\cup_{k=1}^{\infty}S_k^c) \text{ by Lemma II.1.1} \\
&= S_{n_0}\cap(\cap_{k=1}^{\infty}S_k)^c \text{ by Lemma II.1.2 (De Morgan)} \\
&= S_{n_0}\cap\left(\lim_{k\to\infty}S_k\right)^c - S_{n_0}\setminus\lim_{k\to\infty}S_k
\end{aligned}$$

and by the first part of the proof (continuity from below) $\mu(\lim_{k\to\infty}(S_{n_0}\setminus S_k)) = \lim_{k\to\infty}\mu(S_{n_0}\setminus S_k)$. Since $\mu(S_{n_0}\setminus S_k) = \mu(S_{n_0}) - \mu(S_k)$ for $k \geq n_0$ by the "Excision Principle" (Exercise II.2.1(ii); this is where $\mu(S_{n_0}) < \infty$ is needed) we have

$$\mu\left(S_{n_0}\setminus\lim_{k\to\infty}S_k\right) = \mu(S_{n_0}) - \mu\left(\lim_{k\to\infty}S_k\right) = \mu(S_{n_0}) = \lim_{k\to\infty}\mu(S_k). \qquad (*)$$

Theorem II.2.1 (continued 2)

Proof (continued).

$$\begin{aligned}
\lim_{k\to\infty}(S_{n_0} \setminus S_k) &= \cup_{k=1}^{\infty}(S_{n_0} \setminus S_k) = \cup_{k=1}^{\infty}(S_{n_0} \cap S_k^c) \\
&= S_{n_0} \cap (\cup_{k=1}^{\infty} S_k^c) \text{ by Lemma II.1.1} \\
&= S_{n_0} \cap (\cap_{k=1}^{\infty} S_k)^c \text{ by Lemma II.1.2 (De Morgan)} \\
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Theorem II.2.1 (continued 3)

Theorem II.2.1. Every measure is continuous from above and below.

Proof (continued). Therefore

$$\begin{aligned}\mu\left(\lim_{k\to\infty} S_k\right) &= \mu\left(S_{n_0} \setminus \left(S_{n_0} \setminus \lim_{k\to\infty} S_k\right)\right) \\ &\quad \text{since the sequence is decreasing} \\ &= \mu(S_{n_0}) - \mu\left(S_{n_0} \setminus \lim_{k\to\infty} S_k\right) \\ &\quad \text{by the Excision Principle (Exercise II.2.1(ii))} \\ &= \lim_{k\to\infty} \mu(S_k) \text{ by } (*).\end{aligned}$$

□

Theorem II.2.2

Theorem II.2.2. Every finite, nonnegative, additive set function F on a Boolean σ algebra $\mathcal{A}$ and satisfying $F(\varnothing) = 0$, which is either continuous from below at every $R \in \mathcal{A}$ or continuous from above at $\varnothing \in \mathcal{A}$, is necessarily also σ additive or "countably additive" (i.e., μ is a measure).

Proof. Let $S_1, S_2, \ldots$ be any infinite sequence of disjoint sets from $\mathscr{A}$. Then the sequence $R_1, R_2, \ldots$ with $R_n = \dot\cup_{k=1}^n S_k$ is monotonically increasing. Since F is additive by hypothesis then

$$F(R_n) = R\left(\dot\cup_{k=1}^n S_k\right) = \sum_{k=1}^n F(S_k).$$

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If F is continuous from below at $R \in \mathscr{A}$ then

$$F\left(\cup_{k=1}^\infty S_k\right) = F\left(\lim_{n\to\infty} R_n\right) = \lim_{n\to\infty} F(R_n) - \lim_{n\to\infty}\left(\sum_{k=1}^n F(S_k)\right) = \sum_{k=1}^\infty F(S_k).$$

So F is countably additive.

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So F is countably additive.

Theorem II.2.2 (continued)

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Proof (continued). If F is continuous form above at $\varnothing$ then, since $R \setminus R_1, R \setminus R_2, \ldots$ is increasing with

$$\lim_{n\to\infty}(R \setminus R_n) = \lim_{n\to\infty}\left(\dot\cup_{k=1}^{\infty} S_k \setminus \dot\cup_{k=1}^{n} S_k\right) = \lim_{n\to\infty}\left(\dot\cup_{k=n+1}^{\infty} S_k\right) = \varnothing,$$

then we have

$$0 = F(\varnothing) = F\left(\lim_{n\to\infty}(R \setminus R_n)\right) = \lim_{n\to\infty} F(R \setminus R_n) = F(R) = \lim_{n\to\infty} F(R_n).$$

That is, $F\left(\dot\cup_{k=1}^{\infty} S_k\right) = F(R) = \lim_{n\to\infty} F(R_n) = \sum_{k=1}^{\infty} F(S_k)$. □

Lemma II.2.A

Lemma II.2.A. Let $\mathscr{A}$ be a Boolean algebra on set $\mathscr{X}$ and let μ be a measure on $\mathscr{A}$. Define extended real-valued set function

$$\mu^+(R) = \inf \left\{ \sum_{k=1}^{\infty} \mu(S_k) \;\middle|\; R \subset \cup_{k=1}^{\infty} S_k, S_k \in \mathscr{A} \text{ for all } k \in \mathbb{N} \right\}$$

on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$. Sets $S_1, S_2, \ldots \in \mathscr{A}$ such that $R \subset \cup_{k=1}^{\infty} S_k$ are said to *cover* R. For any $R_1, R_2, \ldots \in \mathscr{G}_{\mathscr{X}}$ we have

$$\mu^+\left(\cup_{n=1}^{\infty} R_n\right) \le \sum_{n=1}^{\infty} \mu^+(R_n).$$

Proof. Let $\varepsilon > 0$. For each R_n there is a covering $S_{n1}, S_{n2}, \ldots \in \mathscr{A}$ such that $R_n \subset \cup_{k=1}^{\infty} S_{nk}$ and

$$\mu^+(R_n) \le \sum_{k=1}^{\infty} \mu(S_{nk}) \le \mu^+(R_n) + \frac{\varepsilon}{2^n},$$

by the infimum definition of μ^+.

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Lemma II.2.A (continued)

Proof (continued). Now $\{S_{nk} \mid n, k \in \mathbb{N}\}$ is a countable family of sets in $\mathscr{A}$ and $\cup_{n=1}^{\infty} R_n \subset \cup_{n,k=1}^{\infty} S_{nk}$, so

$$\mu^+\left(\cup_{n=1}^{\infty} R_n\right) \le \sum_{n,k=1}^{\infty} \mu(S_{nk}) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(S_{nk})$$

$$\le \sum_{k=1}^{\infty} \left(\mu^+(R_n) + \frac{\varepsilon}{s^n}\right) = \varepsilon + \sum_{n=1}^{\infty} \mu^+(R_n),$$

Since $\varepsilon > 0$ is arbitrary, the claim follows. □

Lemma II.2.1

Lemma II.2.1. If M is an outer measure on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$ then the class $\mathscr{A}_M$ of all M-measurable sets $S \in \mathscr{G}_{\mathscr{X}}$ is a Boolean σ algebra, and the outer measure M restricted to $\mathscr{A}_M$ is a measure.

Proof. First, we prove $\mathscr{A}_M$ is a Boolean algebra. For any $R \in \mathscr{G}_{\mathscr{X}}$ we have

$$M(R) = M(R \cap \mathscr{X}) = 0 + M(R \cap \mathscr{X}) = M(R \cap \varnothing) + M(R \cap \mathscr{X})$$

and so $\varnothing \in \mathscr{A}_M$. Clearly if S is measurable and satisfies the Carathéodory condition, then S' satisfies the Carathéodory condition and is measurable; that is, if $S \in \mathscr{A}_M$ then $S' \in \mathscr{A}_M$.

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$$M(R) = M(R \cap S_1) + M(R \cap S_1'). \qquad (2.11)$$

With $R \cap S_1, R \cap S_1' \in \mathscr{G}_{\mathscr{X}}$ and the fact that S_2 is measurable then we have

$$M(R \cap S_1) = M((R \cap S_1) \cap S_2) + M((R \cap S_1) \cap S_2')$$

$$\text{and } M(R \cap S_1') = M((R \cap S_1') \cap S_2) + M((R \cap S_1') \cap S_2').$$

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Lemma II.2.1 (continued 1)

Proof (continued). Substituting these into (2.11) we get

$$\begin{aligned} M(R) &= M(R\cap S_1\cap S_2)+M(R\cap S_1\cap S_2')+M(R\cap S_1'\cap S_2) \\ &\quad +M(R\cap S_1'\cap S_2') \\ &= M(R\cap S_1\cap S_2)+M(R\cap S_1\cap S_2')+M(R\cap S_1'\cap S_2) \\ &\quad +M(R\cap (S_1\cup S_2)') \text{ by Lemma II.1.2 (De Morgan)} \end{aligned} \tag{2.14}$$

Now this equation holds for all $R\in\mathscr{G}_{\mathscr{X}}$. So by replacing R by $R\cap(S_1\cap S_2)$ we get

$$M(R\cap(S_1\cap S_2)) = M((R\cap(S_1\cup S_2)\cap S_1\cap S_2)+M((R\cap(S_1\cup S_2)\cap S_1\cap S_2')$$

$$+M((R\cap(S_1\cup S_2))\cap S_1'\cap S_2)+M((R\cap(S_1\cup S_2))\cap(S_1\cup S_2)')$$

$$= M(R\cap S_1\cap S_2)+M(R\cap S_1\cap S_2')+M(R\cap S_1'\cap S_2) \tag{2.15}$$

since. . .

Lemma II.2.1 (continued 1)

Proof (continued). Substituting these into (2.11) we get

$$\begin{aligned} M(R) &= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \\ &\quad + M(R \cap S_1' \cap S_2') \\ &= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \\ &\quad + M(R \cap (S_1 \cup S_2)') \text{ by Lemma II.1.2 (De Morgan)} \qquad (2.14) \end{aligned}$$

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$$M(R \cap (S_1 \cap S_2)) = M((R \cap (S_1 \cup S_2) \cap S_1 \cap S_2) + M((R \cap (S_1 \cup S_2) \cap S_1 \cap S_2')$$

$$+ M((R \cap (S_1 \cup S_2)) \cap S_1' \cap S_2) + M((R \cap (S_1 \cup S_2)) \cap (S_1 \cup S_2)')$$

$$= M(R \cap S_1 \cap S_2) + M(R \cap S_1 \cap S_2') + M(R \cap S_1' \cap S_2) \qquad (2.15)$$

since...

Lemma II.2.1 (continued 2)

Proof (continued). . . . by Lemma II.1.1,

$$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2 = R \cap S_1 \cap S_2 \text{ since } (S_1 \cup S_2) \cap S_1 \cap S_2 = S_1 \cap S_2,$$

$$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2' = R \cap S_1 \cap S_2' \text{ since } (S_1 \cup S_2) \cap S_1 \cap S_2' = S_1 \cap S_2',$$

$$(R \cap (S_1 \cup S_2)) \cap S_1' \cap S_2 = R \cap S_1' \cap S_2 \text{ since } (S_1 \cup S_2) \cap S_1' \cap S_2 = S_1' \cap S_2,$$

$$\text{and } (R \cap (S_1 \cup S_2)) \cap (S_1 \cup S_2)' = \varnothing \text{ since } (S_1 \cup S_2) \cap (S_1 \cup S_2)' = \varnothing.$$

Replacing part of the right hand side of (2.14) with $M(R \cap (S_1 \cup S_2)')$ we get

$$M(R) = M(R \cap (S_1 \cap S_2)) + M(R \cap (S_1 \cup S_2)')$$

for all $R \in \mathscr{G}_{\mathscr{X}}$, so that $S_1 \cup S_2 \in \mathscr{A}_M$. Therefore $\mathscr{A}_M$ is a Boolean algebra.

Lemma II.2.1 (continued 2)

Proof (continued). . . . by Lemma II.1.1,

$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2 = R \cap S_1 \cap S_2$ since $(S_1 \cup S_2) \cap S_1 \cap S_2 = S_1 \cap S_2$,

$(R \cap (S_1 \cup S_2)) \cap S_1 \cap S_2' = R \cap S_1 \cap S_2'$ since $(S_1 \cup S_2) \cap S_1 \cap S_2' = S_1 \cap S_2'$,

$(R \cap (S_1 \cup S_2)) \cap S_1' \cap S_2 = R \cap S_1' \cap S_2$ since $(S_1 \cup S_2) \cap S_1' \cap S_2 = S_1' \cap S_2$,

and $(R \cap (S_1 \cup S_2)) \cap (S_1 \cup S_2)' = \varnothing$ since $(S_1 \cup S_2) \cap (S_1 \cup S_2)' = \varnothing$.

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for all $R \in \mathscr{G}_{\mathscr{X}}$, so that $S_1 \cup S_2 \in \mathscr{A}_M$. Therefore $\mathscr{A}_M$ is a Boolean algebra.

Lemma II.2.1 (continued 3)

Proof (continued). To prove the $\mathscr{A}_M$ is a Boolean σ algebra, we consider a sequence of sets $S_1, S_2, \ldots \in \mathscr{A}_M$. We can assume without loss of generality that the sets are disjoint (or we can use the Boolean algebra properties to create a disjoint sequence from a non-disjoint sequence). With S_1 and S_2 disjoint we have $S_1 \cap S_2' = S_2$ and $S_1' \cap S_2 = S_2$ so that (2.15) becomes

$$M(R \cap (S_1 \cup S_2)) = M(R \cap S_1) + M(R \cap S_2)$$

and inductively we have

$$M\left(R \cap (\cup_{n=1}^{\infty} S_k)\right) = \sum_{k=1}^{n} M(R \cap S_k). \qquad (2.17)$$

Since M is an outer measure and so monotone (by subadditivity and nonnegativity) we have

$$M\left(R \cap (\cup_{k=1}^{n} S_k)'\right) \geq M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right). \qquad (2.18)$$

Lemma II.2.1 (continued 3)

Proof (continued). To prove the $\mathscr{A}_M$ is a Boolean σ algebra, we consider a sequence of sets $S_1, S_2, \ldots \in \mathscr{A}_M$. We can assume without loss of generality that the sets are disjoint (or we can use the Boolean algebra properties to create a disjoint sequence from a non-disjoint sequence). With S_1 and S_2 disjoint we have $S_1 \cap S_2' = S_2$ and $S_1' \cap S_2 = S_2$ so that (2.15) becomes

$$M(R \cap (S_1 \cup S_2)) = M(R \cap S_1) + M(R \cap S_2)$$

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Lemma II.2.1 (continued 4)

Proof (continued). Now $\cup_{k=1}^{n} S_k$ is M measurable since $\mathscr{A}_M$ is a Boolean algebra, so by the Carathéodory splitting condition

$$\begin{aligned} M(R) &= M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{n} S_k)\right) + M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{n} S_k)'\right) \\ &= M\left(\mathbin{\dot\cup}_{k=1}^{n} (R \cap S_k)\right) + M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{n} S_k)'\right) \text{ by Lemma II.1.1} \\ &\geq \sum_{k=1}^{n} M(R \cap S_k) + M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{\infty} S_k)'\right) \text{ by (2.17) and (2.18).} \end{aligned}$$

Since this holds for arbitrary $n \in \mathbb{N}$ then

$$M(R) \geq \sum_{k=1}^{\infty} M(R \cap S_k) + M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{\infty} S_k)'\right). \qquad (*)$$

As an outer measure M is countably subadditive,

$$M\left(R \cap (\mathbin{\dot\cup}_{k=1}^{\infty} S_k)\right) = M\left(\mathbin{\dot\cup}_{k=1}^{\infty} (R \cap S_k)\right) \leq \sum_{k=1}^{\infty} M(R \cap S_k).$$

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Proof (continued). Now $\cup_{k=1}^{n} S_k$ is M measurable since $\mathscr{A}_M$ is a Boolean algebra, so by the Carathéodory splitting condition

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Proof (continued). So from this and $(*)$ we have

$$M(R) \geq \sum_{k=1}^{\infty} M(R \cap S_k) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right)$$
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$$M(R) = M\left((R \cap (\cup_{k=1}^{\infty} S_k)) \cup \left(R \cap (\cup_{k=1}^{\infty} S_k)'\right)\right)$$
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hence $M(R) = M\left(R \cap (\cup_{k=1}^{\infty} S_k)\right) + M\left(R \cap (\cup_{k=1}^{\infty} S_k)'\right)$ for all $R \in \mathscr{G}_{\mathscr{X}}$ so that $\cup_{k=1}^{\infty} S_k$ is M measurable. That is, $\cup_{k=1}^{\infty} S_k \in \mathscr{A}_M$ and so $\mathscr{A}_M$ is a Boolean σ algebra.

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Finally, to prove that M is σ additive on $\mathscr{A}_M$ (and hence is a measure on $\mathscr{A}_M$), take $R = \cup_{k=1}^{\infty} S_k$ is (2,21) to get $M\left(\cup_{k=1}^{\infty} S_n\right) \geq \sum_{k=1}^{\infty} \mu(S_k)$. Countable additivity shows the reverse of this inequality and hence M is σ additive. □

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Lemma II.2.2

Lemma II.2.2. Every set R in the Boolean σ algebra $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is μ^+ measurable. That is, $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$.

Proof. Let $R \in \mathscr{G}_\mathscr{X}$. By the infimum definition of μ^+, for any $\varepsilon > 0$ there is sequence $S_1, S_2, \ldots \in \mathscr{A}$ such that $R \subset \cup_{k=1}^\infty S_k$ and

$$\mu^+(R) \le \sum_{k=1}^\infty \mu(S_k) \le \mu^+(R) + \varepsilon \qquad (2.23)$$

(here, μ is a measure on $\mathscr{A}$ that defines outer measure μ^+).

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$$\sum_{k=1}^\infty \mu(S_k) = \sum_{k=1}^\infty \mu((S_k \cap S) \cup\!\!\!\cdot\, (S_k \cap S')) = \sum_{k=1}^\infty (\mu(S_k \cap S) + \mu(S_k \cap S'))$$

$$= \sum_{k=1}^\infty \mu(S_k \cap S) + \sum_{k=1}^\infty \mu(S_k \cap S') \geq \mu^+(R \cap S) + \mu^+(R \cap S') \qquad (2.24)$$

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Lemma II.2.2 (continued)

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Proof. ...since $R \cap S \subset \cup_{k=1}^{\infty}(S_n \cap S)$ and $R \cap S' \subset \cup_{k=1}^{\infty}(S_k \cap S')$. Combining (2.23) and (2.24) we obtain $\mu^+(R) + \varepsilon \geq \mu^+(R \cap S) + \mu^+(R \cap S')$ or, since $\varepsilon > 0$ is arbitrary, $\mu^+(R) \geq \mu^+(R \cap S) + \mu^+(R \cap S')$. By subadditivity, we can reverse this inequality and conclude that S is measurable; i.e., $S \in \mathscr{A}_{\mu^+}$. Since S is an arbitrary element of $\mathscr{A}$, then $\mathscr{A} \subset \mathscr{A}_{\mu^+}$.

So $\mathscr{A}_{\mu^+}$ is a Boolean σ algebra by Lemma II.2.1, and $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, so we must have $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$, as claimed. □

Lemma II.2.2 (continued)

Lemma II.2.2. Every set R in the Boolean σ algebra $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is μ^+ measurable. That is, $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$.

Proof. ...since $R \cap S \subset \cup_{k=1}^\infty (S_n \cap S)$ and $R \cap S' \subset \cup_{k=1}^\infty (S_k \cap S')$. Combining (2.23) and (2.24) we obtain
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So $\mathscr{A}_{\mu^+}$ is a Boolean σ algebra by Lemma II.2.1, and $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ is the smallest σ algebra containing $\mathscr{A}$, so we must have $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$, as claimed. □

Theorem II.2.3

Theorem II.2.3. Let μ be a measure defined on the Boolean algebra $\mathscr{A}$ of subsets of a given set $\mathscr{X}$. The set function

$$\overline{\mu} = \inf \left\{ \sum_{k=1}^{\infty} \mu(S_k) \;\middle|\; R \subset \cup_{k=1}^{\infty} S_k, S_k \in \mathscr{A} \text{ for all } n \in \mathbb{N} \right\}$$

for $R \in \mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$, is a measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A} : \overline{\mu}(S) = \mu(S)$ for all $S \in \mathscr{A}$. If μ is a σ finite measure, then $\overline{\mu}$ is also σ finite, and $\overline{\mu}$ is the only measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A}$. The measure $\overline{\mu}$ is called the *extension* of μ.

Proof.

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for $R \in \mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$, is a measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A}$: $\overline{\mu}(S) = \mu(S)$ for all $S \in \mathscr{A}$. If μ is a σ finite measure, then $\overline{\mu}$ is also σ finite, and $\overline{\mu}$ is the only measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A}$. The measure $\overline{\mu}$ is called the *extension* of μ.

Proof.