Chapter 9. Compact Operators
9.5. Compact Self Adjoint Operators on Hilbert Spaces—Proofs of Theorems
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Proposition 9.17. If $M$ is invariant for compact, self adjoint operator $T$ on a Hilbert space then $M^\perp$ is invariant for $T$. Moreover, the restrictions of $T$ to both $M$ and $M^\perp$ are also self adjoint.

Proof. For all $x \in M$ and $y \in M^\perp$ we have
\[
\langle Ty, x \rangle = \langle y, T^*x \rangle = \langle y, Tx \rangle = 0 \text{ since } Tx \in M \text{ because } M \text{ is invariant under } T. \text{ Therefore } Ty \in M^\perp. \text{ Since } y \text{ is an arbitrary element of } M^\perp \text{ then } M^\perp \text{ is invariant under } T.
**Proposition 9.17.** If $M$ is invariant for compact, self adjoint operator $T$ on a Hilbert space then $M^\perp$ is invariant for $T$. Moreover, the restrictions of $T$ to both $M$ and $M^\perp$ are also self adjoint.

**Proof.** For all $x \in M$ and $y \in M^\perp$ we have 

$$\langle Ty, x \rangle = \langle y, T^* x \rangle = \langle y, Tx \rangle = 0$$

since $Tx \in M$ because $M$ is invariant under $T$. Therefore $Ty \in M^\perp$. Since $y$ is an arbitrary element of $M^\perp$ then $M^\perp$ is invariant under $T$.

Since $T$ is self adjoint on $H$ and $M$ and $M^\perp$ are invariant under $T$, the $T$ restricted to $M$ and $M^\perp$ is self adjoint (that is, $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in M$ and for all $x, y \in M^\perp$).
Proposition 9.17

Proposition 9.17. If $M$ is invariant for compact, self adjoint operator $T$ on a Hilbert space then $M^\perp$ is invariant for $T$. Moreover, the restrictions of $T$ to both $M$ and $M^\perp$ are also self adjoint.

Proof. For all $x \in M$ and $y \in M^\perp$ we have

\[ \langle Ty, x \rangle = \langle y, T^* x \rangle = \langle y, Tx \rangle = 0 \]

since $Tx \in M$ because $M$ is invariant under $T$. Therefore $Ty \in M^\perp$. Since $y$ is an arbitrary element of $M^\perp$ then $M^\perp$ is invariant under $T$.

Since $T$ is self adjoint on $H$ and $M$ and $M^\perp$ are invariant under $T$, the $T$ restricted to $M$ and $M^\perp$ is self adjoint (that is, $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in M$ and for all $x, y \in M^\perp$). \qed

Let $T$ be a compact, self adjoint operator on a Hilbert space $H$. There is a sequence (either finite or countably infinite) of mutually orthogonal closed subspaces $(M_n)$ whose closed linear span is all of $H$. There is a corresponding sequence $(\lambda_n)$ of real numbers which if countably infinite converges to 0. For all $n$ and $x \in M_n$, we have $Tx = \lambda_n x$. Moreover, if $\lambda_n \neq 0$ then $M_n$ is finite dimensional.

Proof. Let $\{\lambda_n\}$ be the set of distinct eigenvalues of $T$. Notice that each $\lambda_n$ is real by Proposition 8.18(a). Let $M_n$ be the eigenspace for $\lambda_n$ (so $Tx = \lambda_n x$ for all $x \in M_n$). Let $K$ be the closed span of all these eigenspaces: $K = \overline{\text{span}}\{M_n \mid n \text{ is an index for } \{\lambda_n\}\}$. 
Theorem 9.18

Let $T$ be a compact, self adjoint operator on a Hilbert space $H$. There is a sequence (either finite or countably infinite) of mutually orthogonal closed subspaces $(M_n)$ whose closed linear span is all of $H$. There is a corresponding sequence $(\lambda_n)$ of real numbers which if countably infinite converges to 0. For all $n$ and $x \in M_n$, we have $Tx = \lambda_n x$. Moreover, if $\lambda_n \neq 0$ then $M_n$ is finite dimensional.

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Theorem 9.18 (continued 1)

**Proof (continued).** If 0 is an eigenvalue then the corresponding eigenspace is the nullspace \( N(T) \) which is closed since \( T \) is continuous. Also by Theorem 9.16, if there are a countably infinite number of eigenvalues then they converge to 0.

Now we show the final claim that \( K = H \). Since each \( M_n \) is an eigenspace for \( \lambda_n \), then \( M_n \) is invariant under \( T \). So \( K \) is invariant under \( T \) (since each \( M_n \) is invariant and \( T \) is continuous on \( H \) by Theorem 2.6). Then by Proposition 9.17, \( K^\perp \) is invariant under \( T \).
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Proof (continued). If 0 is an eigenvalue then the corresponding eigenspace is the nullspace $N(T)$ which is closed since $T$ is continuous. Also by Theorem 9.16, if there are a countably infinite number of eigenvalues then they converge to 0.

Now we show the final claim that $K = H$. Since each $M_n$ is an eigenspace for $\lambda_n$, then $M_n$ is invariant under $T$. So $K$ is invariant under $T$ (since each $M_n$ is invariant and $T$ is continuous on $H$ by Theorem 2.6). Then by Proposition 9.17, $K^\perp$ is invariant under $T$. ASSUME $K^\perp \neq 0$. Let $T_1$ denote the restriction of $T$ to $K^\perp$. Since a subset of any relatively compact set is relatively compact (the closure of the subset is a closed subset of the [compact] closure of the superset and so is compact; see page 18), from the definition of “compact operator” we have that the restriction of a compact operator must be compact.
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Proof (continued). If 0 is an eigenvalue then the corresponding eigenspace is the nullspace $N(T)$ which is closed since $T$ is continuous. Also by Theorem 9.16, if there are a countably infinite number of eigenvalues then they converge to 0.

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**Proof (continued).** If 0 is an eigenvalue then the corresponding eigenspace is the nullspace $N(T)$ which is closed since $T$ is continuous. Also by Theorem 9.16, if there are a countably infinite number of eigenvalues then they converge to 0.

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Theorem 9.18 (continued 2)

**Proof (continued).** But then 0 is an eigenvalue for $T$ and so $x$ is in the eigenspace associated with eigenvalue 0 (it’s one of the $M_n$’s) and so $x \in K$, a contradiction since $K \cap K^\perp = \{0\}$ by the Projection Theorem [Theorem 4.14]); so $T_1$ is not the zero operator on $K^\perp$. By Proposition 8.21 either $\|T_1\|$ or $-\|T_1\|$ is in $\sigma(T_1)$. Since the value is nonzero, by Theorem 9.16 it is an eigenvalue of $T_1$, and so also is an eigenvalue of $T$. But then the corresponding (nonzero) eigenvectors is in both $K$ and $dK^\perp$, a CONTRADICTION (again, by the Projection Theorem). So the assumption that $K^\perp \neq \{0\}$ is false, and $K^\perp = \{0\}$. That is, $H = K = \text{span}\{M_n \mid n \text{ is an index for } \{\lambda_n\}\}$.
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**Proof (continued).** But then 0 is an eigenvalue for $T$ and so $x$ is in the eigenspace associated with eigenvalue 0 (it’s one of the $M_n$’s) and so $x \in K$, a contradiction since $K \cap K^\perp = \{0\}$ by the Projection Theorem [Theorem 4.14]); so $T_1$ is not the zero operator on $K^\perp$. By Proposition 8.21 either $\|T_1\|$ or $-\|T_1\|$ is in $\sigma(T_1)$. Since the value is nonzero, by Theorem 9.16 it is an eigenvalue of $T_1$, and so also is an eigenvalue of $T$. But then the corresponding (nonzero) eigenvectors is in both $K$ an d $K^\perp$, a CONTRADICTION (again, by the Projection Theorem). So the assumption that $K^\perp \neq \{0\}$ is false, and $K^\perp = \{0\}$. That is, $H = K = \text{span}\{M_n \mid n \text{ is an index for } \{\lambda_n\}\}$. 

$\square$
Theorem 9.19

Theorem 9.19. For $T$ a compact, self adjoint operator on Hilbert space $H$, $T = \sum_n \lambda_n E_{\lambda_n}$ in which $E_{\lambda_n}$ is the projection onto $M_n$ where $M_n$ is the eigenspace associated with $\lambda_n$.

Proof. If $T$ only has a finite number of eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $H$ is the closed linear space of $M_1, M_2, \ldots, M_n$; that is, $H = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ (since there are only finitely many $M_k$’s). But then for any $x \in H$, say $x = x_1 + x_2 + \cdots + x_n$ where $x_k \in M_k$, we have

$$T(x) = T(x_1 + x_2 + \cdots + x_n) = T(x_1) + T(x_2) + \cdots + T(x_n)$$

$$= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

$$= \lambda_1 E_{\lambda_1}(x) + \lambda_2 E_{\lambda_2}(x) + \cdots + \lambda_n E_{\lambda_n}(x)$$

$$= \sum_k \lambda_k E_{\lambda_k},$$

as claimed.
Theorem 9.19. For $T$ a compact, self adjoint operator on Hilbert space $H$, $T = \sum_n \lambda_n E_{\lambda_n}$ in which $E_{\lambda_n}$ is the projection onto $M_n$ where $M_n$ is the eigenspace associated with $\lambda_n$.

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$$

$$
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as claimed.
Theorem 9.19 (continued 1)

Proof (continued). If $T$ has an infinite number of eigenvalues then, by the Spectral Theorem for Compact, Self Adjoint Operators (Theorem 9.18), the eigenvalues form a (countable) sequence $(\lambda_n)$ with $(\lambda_n) \to 0$. Let $\varepsilon > 0$. Let $S_n = \sum_{k=1}^{n} \lambda_k E_{\lambda_k}$ (the $n$th partial sum) and let $T_n = T - S_n$ (the “tail”). Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Recall that a projection $P$ satisfies (by definition) $P = P^*$ and $P^2 = P$, so the projection $E_{\lambda_k}$ is self adjoint. By Proposition 9.10(a,b), $T_n$ is self adjoint for all $n \in \mathbb{N}$. 
Proof (continued). If $T$ has an infinite number of eigenvalues then, by the Spectral Theorem for Compact, Self Adjoint Operators (Theorem 9.18), the eigenvalues form a (countable) sequence $(\lambda_n)$ with $(\lambda_n) \to 0$. Let $\varepsilon > 0$. Let $S_n = \sum_{k=1}^{n} \lambda_k E_{\lambda_k}$ (the $n$th partial sum) and let $T_n = T - S_n$ (the “tail”). Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Recall that a projection $P$ satisfies (by definition) $P = P^*$ and $P^2 = P$, so the projection $E_{\lambda_k}$ is self adjoint. By Proposition 9.10(a,b), $T_n$ is self adjoint for all $n \in \mathbb{N}$. For $x \in M_k$ where $1 \leq k \leq n$ we have

$$T_n x = (T - S_n) x = T x - \sum_{k=1}^{n} \lambda_k E_{\lambda_k} x$$

$$= T x - \lambda_k E_{\lambda_k} x \text{ since } E_{\lambda_i} x = 0 \text{ for } i \neq k$$

$$= \lambda_k x - \lambda_k x \text{ since } x \text{ is in eigenspace } M_k \text{ of } \lambda_k$$

$$= 0.$$

So $T_n$ is 0 on $K = \overline{\text{span}} \{M_1, M_2, \ldots, M_n\}$ because $T_n$ is continuous (since it is bounded; see Theorem 2.6).
Theorem 9.19 (continued 1)

**Proof (continued).** If $T$ has an infinite number of eigenvalues then, by the Spectral Theorem for Compact, Self Adjoint Operators (Theorem 9.18), the eigenvalues form a (countable) sequence $(\lambda_n)$ with $(\lambda_n) \to 0$. Let $\varepsilon > 0$. Let $S_n = \sum_{k=1}^{n} \lambda_k E_{\lambda_k}$ (the $n$th partial sum) and let $T_n = T - S_n$ (the “tail”). Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Recall that a projection $P$ satisfies (by definition) $P = P^*$ and $P^2 = P$, so the projection $E_{\lambda_k}$ is self adjoint. By Proposition 9.10(a,b), $T_n$ is self adjoint for all $n \in \mathbb{N}$. For $x \in M_k$ where $1 \leq k \leq n$ we have

$$T_n x = (T - S_n)x = Tx - \sum_{k=1}^{n} \lambda_k E_{\lambda_k}x$$

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$$= \lambda_k x - \lambda_k x \text{ since } x \text{ is in eigenspace } M_k \text{ of } \lambda_k$$

$$= 0.$$

So $T_n$ is 0 on $K = \text{span}\{M_1, M_2, \ldots, M_n\}$ because $T_n$ is continuous (since it is bounded; see Theorem 2.6).
Theorem 9.19 (continued 2)

Proof (continued). For $x \in K^\perp = \overline{\text{span}}\{M_1, M_2, \ldots, M_n\}^\perp$ we have

$$T_nx = (T - S_n)x = Tx - \sum_{k=1}^{n} \lambda_k E_{\lambda_k}x = Tx.$$ 

Next, if $x$ is an eigenvector of $T_n$ where $n \geq N$ with corresponding eigenvalue $\lambda$ then

$$\lambda x = T_nx = T_x(x_K + x_{K^\perp}) = T_n x_{K^\perp} = Tx_{K^\perp}$$

where $x_K \in K$ and $x_{K^\perp} \in K^\perp$. If $x \in K$ then $x_{K^\perp} = 0$. But then $\lambda x = Tx_{K^\perp} = T0 = 0$ and so $\lambda = 0$. 


Theorem 9.19 (continued 2)

Proof (continued). For $x \in K^\bot = \overline{\text{span}}\{M_1, M_2, \ldots, M_n\}^\bot$ we have

$$T_nx = (T - S_n)x = Tx - \sum_{k=1}^{n} \lambda_k E_{\lambda_k} x = Tx.$$ 

Next, if $x$ is an eigenvector of $T_n$ where $n \geq N$ with corresponding eigenvalue $\lambda$ then

$$\lambda x = T_nx = Tx(x_K + x_{K^\bot}) = T_nx_{K^\bot} = Tx_{K^\bot}$$

where $x_K \in K$ and $x_{K^\bot} \in K^\bot$. If $x \in K$ then $x_{K^\bot} = 0$. But then $\lambda x = Tx_{K^\bot} = T0 = 0$ and so $\lambda = 0$. If $x \notin K$ then $x_{K^\bot} \neq 0$. Since $x \in H$ and $H$ is the closed linear span of the $M_n$'s, then $x = \sum_{k=1}^{\infty} a_k x_k$ where $x_k \in M_k$. 


Theorem 9.19 (continued 2)

Proof (continued). For \( x \in K^\perp = \overline{\text{span}}\{M_1, M_2, \ldots, M_n\}^\perp \) we have

\[
T_nx = (T - S_n)x = Tx - \sum_{k=1}^{n} \lambda_k E_{\lambda_k}x = Tx.
\]

Next, if \( x \) is an eigenvector of \( T_n \) where \( n \geq N \) with corresponding eigenvalue \( \lambda \) then

\[
\lambda x = T_nx = T_x(x_K + x_{K^\perp}) = T_nx_{K^\perp} = Tx_{K^\perp}
\]

where \( x_K \in K \) and \( x_{K^\perp} \in K^\perp \). If \( x \in K \) then \( x_{K^\perp} = 0 \). But then \( \lambda x = Tx_{K^\perp} = T0 = 0 \) and so \( \lambda = 0 \). If \( x \notin K \) then \( x_{K^\perp} \neq 0 \). Since \( x \in H \) and \( H \) is the closed linear span of the \( M_n \)'s, then \( x = \sum_{k=1}^{\infty} a_k x_k \) where \( x_k \in M_k \).
Theorem 9.19 (continued 3)

Proof (continued). Since $T_n$ is continuous then

$$T_n x = T_n \left( \sum_{k=1}^{\infty} a_k x_k \right) = \sum_{k=1}^{\infty} a_k T_n x_k$$

$$= \sum_{k=n+1}^{\infty} a_k T_n x_k \text{ since } T_n \text{ is 0 on } M_1, M_2, \ldots, M_n$$

$$= \sum_{k=n+1}^{\infty} a_k T x_k \text{ since all such } x_k \in K^\perp$$

$$= \sum_{k=n+1}^{\infty} a_k \lambda_k x_k \text{ since } x_k \in M_k$$

$$= \lambda x = \lambda \sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} a_k \lambda x_k.$$ 

So $a_1 = a_2 = \cdots = a_n = 0$ and $a_k \lambda = a_k \lambda_k$ for $k \geq n + 1.$
Theorem 9.19 (continued 4)

**Proof (continued).** Since $x_K \perp \neq 0$ then some $a_k \neq 0$ for $k \geq n + 1$ and then $\lambda = \lambda_k$ for some $k \geq n + 1$. Since such $\lambda_k$ satisfies $|\lambda_k| < \varepsilon$, then $|\lambda| < \varepsilon$. Therefore, any eigenvalue $\lambda$ of $T_n$ satisfies $|\lambda| < \varepsilon$ when $n > N$.

Since $T_n$ is compact, by Theorem 9.16 the nonzero elements of the spectrum are eigenvalues and so the spectral radius satisfies $r(T_n) < \varepsilon$. Since $T_n$ is self adjoint, then $T_n = T_n^*$ and so $T_n T_n^* = T_n^* T_n$, so $T_n$ is (by definition) normal. By Theorem 8.23, $\|T_n\| = r(T_n) < \varepsilon$. That is, for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ we have $\|T_n\| < \varepsilon$. So $(T_n) \to 0$ or $(T - S_n) \to 0$ or $S_n \to T$. 
Theorem 9.19 (continued 4)

Proof (continued). Since \( x_{K^\perp} \neq 0 \) then some \( a_k \neq 0 \) for \( k \geq n + 1 \) and then \( \lambda = \lambda_k \) for some \( k \geq n + 1 \). Since such \( \lambda_k \) satisfies \( |\lambda_k| < \varepsilon \), then \( |\lambda| < \varepsilon \). Therefore, any eigenvalue \( \lambda \) of \( T_n \) satisfies \( |\lambda| < \varepsilon \) when \( n > N \).

Since \( T_n \) is compact, by Theorem 9.16 the nonzero elements of the spectrum are eigenvalues and so the spectral radius satisfies \( r(T_n) < \varepsilon \).

Since \( T_n \) is self adjoint, then \( T_n = T_n^* \) and so \( T_n T_n^* = T_n^* T_n \), so \( T_n \) is (by definition) normal. By Theorem 8.23, \( \|T_n\| = r(T_n) < \varepsilon \). That is, for given \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for \( n > N \) we have \( \|T_n\| < \varepsilon \). So \( (T_n) \to 0 \) or \( (T - S_n) \to 0 \) or \( S_n \to T \). That is,

\[
T = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k E_{\lambda_k} = \sum_{n=1}^{\infty} \lambda_n E_{\lambda_n}.
\]
Theorem 9.19 (continued 4)

Proof (continued). Since $x_{K^\perp} \neq 0$ then some $a_k \neq 0$ for $k \geq n + 1$ and then $\lambda = \lambda_k$ for some $k \geq n + 1$. Since such $\lambda_k$ satisfies $|\lambda_k| < \varepsilon$, then $|\lambda| < \varepsilon$. Therefore, any eigenvalue $\lambda$ of $T_n$ satisfies $|\lambda| < \varepsilon$ when $n > N$. Since $T_n$ is compact, by Theorem 9.16 the nonzero elements of the spectrum are eigenvalues and so the spectral radius satisfies $r(T_n) < \varepsilon$.

Since $T_n$ is self adjoint, then $T_n = T_n^*$ and so $T_n T_n^* = T_n^* T_n$, so $T_n$ is (by definition) normal. By Theorem 8.23, $\|T_n\| = r(T_n) < \varepsilon$. That is, for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ we have $\|T_n\| < \varepsilon$. So $(T_n) \to 0$ or $(T - S_n) \to 0$ or $S_n \to T$. That is,

$$ T = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k E_{\lambda_k} = \sum_{n=1}^{\infty} \lambda_n E_{\lambda_n}. $$
Theorem 9.20. A compact, self adjoint operator $T$ on a separable Hilbert space is unitarily equivalent to a multiplication operator $M_f$ on $\ell^2$.

Proof. Choose an orthonormal basis of eigenvectors $(e_n)$ and corresponding eigenvalues $(\mu_n)$ such that $T(x) = \sum_k \mu_k \langle x, e_k \rangle e_k$, as described in the note above.
Theorem 9.20

Theorem 9.20. A compact, self adjoint operator $T$ on a separable Hilbert space is unitarily equivalent to a multiplication operator $M_f$ on $\ell^2$.

Proof. Choose an orthonormal basis of eigenvectors $(e_n)$ and corresponding eigenvalues $(\mu_n)$ such that $T(x) = \sum_k \mu_k \langle x, e_k \rangle e_k$, as described in the note above. Let $U : \ell^2 \to H$ be defined as $U(\delta_n) = e_n$ where $\delta_n$ is the $n$th standard vector for $\ell^2$. Then by Theorem 4.19 (see the proof of it) $U$ is an isometric isomorphism (and so is bijective).
Theorem 9.20. A compact, self adjoint operator $T$ on a separable Hilbert space is unitarily equivalent to a multiplication operator $M_f$ on $\ell^2$.

Proof. Choose an orthonormal basis of eigenvectors $(e_n)$ and corresponding eigenvalues $(\mu_n)$ such that $T(x) = \sum_k \mu_k \langle x, e_k \rangle e_k$, as described in the note above. Let $U : \ell^2 \to H$ be defined as $U(\delta_n) = e_n$ where $\delta_n$ is the $n$th standard vector for $\ell^2$. Then by Theorem 4.19 (see the proof of it) $U$ is an isometric isomorphism (and so is bijective). Now

$$U^{-1}TU(\delta_n) = U^{-1}T(e_n) = U^{-1}(\mu_n \langle e_n, e_n \rangle e_n)$$

$$= U^{-1}(\mu_n e_n) = \mu_n U^{-1}(e_n) = \mu_n \delta_n.$$ 

So with $f(x) = \mu_n$, then the multiplication operator $M_f$ maps $\delta_n$ to $\mu_n \delta_n$. 


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**Proof.** Choose an orthonormal basis of eigenvectors $(e_n)$ and corresponding eigenvalues $(\mu_n)$ such that $T(x) = \sum_k \mu_k \langle x, e_k \rangle e_k$, as described in the note above. Let $U : \ell^2 \to H$ be defined as $U(\delta_n) = e_n$ where $\delta_n$ is the $n$th standard vector for $\ell^2$. Then by Theorem 4.19 (see the proof of it) $U$ is an isometric isomorphism (and so is bijective). Now

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So with $f(x) = \mu_n$, then the multiplication operator $M_f$ maps $\delta_n$ to $\mu_n \delta_n$. Since $U^{-1}TU$ and $M_f$ agree on the basis $\{\delta_n\}_{n=1}^\infty$ of $\ell^2$, then $U^{-1}TU$ and $M_f$ are equal on $\ell^2$. So $M_f$ and $T$ are (by definition) unitarily equivalent.
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