Introduction to Functional Analysis, MATH 5740, Summer 2017

Homework 5, Chapter 2, Sections 6 and 8

Due Friday, June 23 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

2.22. Prove that for linear space X with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, if for all $r > 0$ there is $s > 0$ such that $B_2(0;s) \subseteq B_1(0;r)$ (where B_i denotes a ball with respect to $\|\cdot\|_i$), then $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. HINT: Use the definition of weaker norm.

2.25. Let X be the space of all continuous functions f on $[0,1]$ such that $f(0) = 0$, with the sup norm. Let $Y = \{f \in X \mid \int_0^1 f(t)\,dt = 0\} \subseteq X$. Prove that for any $g \in X$ and positive integer $n \in \mathbb{N}$, there is a constant $\alpha \in \mathbb{R}$ such that $g - \alpha t^{1/n} \in Y$. Conclude that perpendiculars to closed proper subspaces need not exist. HINT: Show that Y is closed (you will need uniform convergence under the sup norm for this). Assume g is a perpendicular to Y. Use properties of $\|g\|_\infty$ and $\int_0^1 g(t)\,dt$ to conclude g is the 0 function.

2.29 Let (x_n) be a sequence in a finite dimensional space X, and let $x \in X$ be such that for all $f \in \mathcal{B}(X, \mathbb{F})$ the sequence $f(x_n)$ converges to $f(x)$. Prove that (x_n) converges to x. HINT: Suppose $\{b_1, b_2, \ldots, b_n\}$ is a basis of unit vectors for X. Consider $f_m \in \mathcal{B}(X, \mathbb{F})$ defined as

$$f_m(b_k) = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

(then since f_m is linear, this defines f_m on all of X). Use an ε argument to show that $(x_n) \to x$ with respect to the sup norm. (Notice Theorem 2.31(b).)