# Introduction to Functional Analysis, MATH 5740, Summer 2017

## Homework 6, Chapter 2, Sections 11 and 12

Due Friday, June 30 at 11:20

**Write in complete sentences!!!** *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

**2.41.** **(c)** Prove that a space with a Schauder basis is separable. HINT: Let $A$ be the set of all finite linear combinations of elements of the Schauder basis with rational coefficients (or rational complex coefficients). Then $A$ is countable. For $x \in X$ and $\varepsilon > 0$, find $a \in A$ such that $a \in B(x;\varepsilon)$. Construct $a$ by taking $N$ sufficiently large such that $\|\sum_{k=1}^N \alpha_k b_k - x\| < \varepsilon/2$ (where $x = \sum_{k=1}^\infty \alpha_k b_k$ where $\{b_1, b_2, \ldots\}$ is the Schauder basis). Then approximate the first $N$ parts of $x$ with rational coefficients (within $\varepsilon/2$).

**2.42.** Prove that $T \in \mathcal{B}(X,Y)$ is a contraction if and only if $\|T\| < 1$.

**2.44.** Consider the metric space $[1,\infty)$ with the usual absolute value metric, and consider the mapping defined by $T(x) = x + 1/x$. Prove that $d(T(x),T(y)) < d(x,y)$, but there is no fixed point.

**3.2.** In $c_{00}$ find a nested sequence of closed sets with diameters approaching zero that has an empty intersection. HINT: Define $x_n = (1 + 1/n, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$ and $x_{-n} = (1 - 1/n, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$. Let $E_N = \{x_n, x_{-n} \mid n \in \mathbb{N}, n \geq N\}$. Show that the $E_N$ are nested, closed, and $\operatorname{diam}(E_N) \to 0$, but $\cap E_N = \varnothing$.