Introduction to Functional Analysis, MATH 5740, Summer 2017

Homework 8, Sections 4.2 and 4.2, Solutions

Due Friday, July 7 at 11:20

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

4.4(a). Let (x_i) be an orthonormal set in a Hilbert space H. Prove Bessel's Inequality: For any $z \in H$, $\|z\|^2 \geq \sum_{i=1}^{\infty} |\langle z, x_i\rangle|^2$, and the equality holds for all z if and only if (x_n) is an orthonormal basis. HINT: Consider $\|z - \sum_{i=1}^{n} \langle z, x_i\rangle x_i\|^2$. When dealing with equality, use Theorem 4.17 and Theorem 4.14.

4.5. Let (x_n) be a sequence in a Hilbert space H such that for all $y \in H$, $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ and such that $\|x_n\|$ converges to $\|x\|$. Prove that (x_n) converges to x. HINT: Use an ε argument with $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ to show that $(x_n) \to x$.

4.16. Let $z_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ for $n \in \mathbb{Z}$. Assume that (z_n) is an orthonormal basis for $L^2[-\pi, \pi]$. Consider $f(x) = x$ on $[-\pi, \pi]$. HINT: The Fundamental Theorem of Calculus holds for all of the integrals in this problem. Use Theorem 4.17(a) and (b).

(a) Calculate $\|f\|_2$.

(b) Calculate $\langle f, z_n\rangle$ for all $n \in \mathbb{Z}$.

(c) **(Bonus)** Use the answers to Parts (a) and (b) to prove the identity $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.