Fundamentals of Functional Analysis

MATH 5740, Summer 2021

Homework 5, Chapter 2

Due Thursday, June 24 at 1:00

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, notes, or hypotheses.

2.20. Consider the sup norm on c_0 and c_{00}.

(a) Prove that in c_0 the series $\sum_{k=1}^{\infty} \delta_k/k$ is convergent but not absolutely convergent.

2.21. Consider the interval $X = [1, \infty)$ with the metric $d(x,y) = |x - y|$ and the metric $d_1(x,y) = |1/x - 1/y|$. Show the two metrics are equivalent in the sense that both have the same convergent sequences. HINT: Suppose $(x_n) \to y$ with respect to one of the metrics. Use an ε-argument, where ε is based on the value of y, to show convergence with respect to the other metric.

BONUS: In Exercise 2.21, show that equivalent metrics do not necessarily preserve boundedness or completeness.

2.22. Prove that for linear space X with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, if for all $r > 0$ there is $s > 0$ such that $B_2(0; s) \subseteq B_1(0; r)$ (where B_i denotes a ball with respect to $\|\cdot\|_i$), then $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. HINT: Use the definition of weaker norm.