Fundamentals of Functional Analysis

MATH 5740, Summer 2021

Homework 7, Chapter 2

Due Thursday, July 1 at 1:00

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, notes, or hypotheses.

2.27. Prove that if a normed linear space X has a complete subspace M such that X/M is complete, then X is complete. HINT: Use the completeness of M to show M is closed. Choose an absolutely summable series in X, $\sum_{k=1}^{\infty} \|x_k\|$. Show $\sum_{k=1}^{\infty} \|\overline{x}_k\|$ is absolutely summable. Use this WITH DETAILS to show that $\sum_{k=1}^{\infty} x_k$ is summable and use Theorem 2.12.

2.28 Consider the space of continuous functions on $[0, 2]$ with the norm $\|f\| = \left(\int_0^2 |f(t)|^2\, dt\right)^{1/2}$. Notice that this is a subspace of $L^2([0, 2])$.

(a) Prove that the sequence (f_n) in which

$$f_n(t) = \begin{cases} 0 & t \in [0, 1) \\ n(t-1) & t \in [1, 1 + 1/n) \\ 1 & t \in [1 + 1/n, 2] \end{cases}$$

is Cauchy but does not converge in this space. NOTE: This shows that the space is not complete.

2.29 Let (x_n) be a sequence in a finite dimensional space X, and let $x \in X$ be such that for all $f \in \mathcal{B}(X, \mathbb{F})$ the sequence $f(x_n)$ converges to $f(x)$. Prove that (x_n) converges to x. HINT: Suppose $\{b_1, b_2, \ldots, b_n\}$ is a basis of unit vectors for X. Consider $f_m \in \mathcal{B}(X, \mathbb{F})$ defined as

$$f_m(b_k) = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

(then since f_m is linear, this defines f_m on all of X). Use an ε argument to show that $(x_n) \to x$ with respect to the sup norm. (Notice Theorem 2.31(b).)