Fundamentals of Functional Analysis

MATH 5740, Summer 2021

Homework 8, Chapters 3 and 4

Due **Sunday, July 11 at 5:00 p.m. (REALLY!!!)**

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, notes, or hypotheses.

3.4. Let X be a linear space that has a countably infinite Hamel basis. Prove that there is no norm on X that will make it into a Banach space. HINT: Let $B = \{b_1, b_2, \ldots\}$ be a Hamel basis. Define $V_n = \operatorname{span}\{b_1, b_2, \ldots, b_n\}$. Show $X = \cup_{n=1}^{\infty} V_n$ and use Corollary 3.3.

4.1. Prove that in any inner product space, two elements x and y are orthogonal if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{C}$.

4.5. Let (x_n) be a sequence in a Hilbert space H such that for all $y \in H$, $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ and such that $\|x_n\|$ converges to $\|x\|$. Prove that (x_n) converges to x. HINT: Use an ε argument with $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ to show that $(x_n) \to x$.