Chapter 1. Normed Vector Spaces

Section 1.2. Vector Spaces

Note. We will consider vector spaces over the scalar fields $\mathbb{R}$ (the reals) and $\mathbb{C}$ (the complex numbers). We denote the scalar field as F and the vector set as E.

Definition 1.2.1. By a *vector space* we mean a nonempty set E with two operations: (1) a mapping $(x, y) \mapsto x + y$ from $E \times E$ into E called *addition*, and (2) a mapping $(\lambda, x) \mapsto \lambda x$ from $F \times E$ into E called *multiplication by scalars*; we require that for all $x, y \in E$ and for all $\alpha, \beta \in F$:

(a) $x + y = y + x$,

(b) $(x + y) + z = z + (y + z)$,

(c) there exists $z \in E$ such that $x + z = y$,

(d) $\alpha(\beta x) = (\alpha\beta)x$,

(e) $(\alpha + \beta)x = \alpha x + \beta y$,

(f) $\alpha(x + y) = \alpha x + \alpha y$, and

(g) $ax = x$.

Note. Property (c) implies the existence of an additive identity vector, called the *zero vector*. Properties (a) and (b) imply this vector is unique.

Note. We can show:

1. If $\lambda \neq 0$ and $\lambda x = 0$ then $x = 0$.

2. If $x \neq 0$ and $\lambda x = 0$ then $\lambda = 0$.

3. $0x = 0$ and $(-1)x = -x$.

Note. Examples of vector spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$. The set of all analytic functions of a complex variable (over some given set) is a vector space where the vectors are functions.

Definition. A subset E_1 of a vector space E is a *subspace* if for every $\alpha, \beta \in F$ and $x, y \in E_1$, the vector $\alpha x + \beta y$ is in E_1.

Example 1.2.5. Denote by ℓ^p, for $p \geq 1$, the space of all infinite sequences, we need only show ℓ^p is closed under scalar multiplication and vector addition. We shall do so soon.

Theorem 1.2.1. Hölder's Inequality.
Let $p > 1$, $q > 1$, and $1/p + 1/q = 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}.$$

Note. The following result is fundamental in establishing that ℓ^p, $p \geq 1$, is a vector space.

Theorem 1.2.2. Minkowski's Inequality.

Let $p \geq 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$ we have

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}.$$

Note. Minkowski's Inequality is the ℓ^p space version of the Triangle Inequality.

Corollary 1.2.A. The ℓ^p spaces for $p \geq 1$ are vector spaces.

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