Section 1.3. Linear Independence, Basis, Dimension

Note. We give several definitions related to vector spaces.

Definition 1.3.1. Let E be a vector space and let $x_1, x_2, \ldots, x_k \in E$. A vector $x \in E$ is a *linear combination* of these vectors if $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Definition 1.3.2. A finite collection of vectors $\{x_1, x_2, \ldots, x_k\}$ if *linearly independent* if $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. An arbitrary (perhaps uncountable) collection of vectors is linearly independent if every finite subcollection is linearly independent. A collection of vectors which is not linearly independent is *linearly dependent*.

Note. Linear independence may depend on the scalar field. For example, 1 and i (as vectors) are linearly independent over the scalar field $\mathbb{R}$, but not over the scalar field $\mathbb{C}$.

Definition. Let $\mathcal{A}$ be a subset of a vector space E. The set of all finite linear combinations of elements of $\mathcal{A}$ is the *span* of $\mathcal{A}$.

Definition 1.3.3. A set of vectors $\mathcal{B} \subset E$ is called a *basis of* E (or a base of E) if $\mathcal{B}$ is linearly independent and $\text{span}(\mathcal{B}) = E$. If a vector space has a finite basis then it is *finite dimensional*, otherwise it is infinite dimensional.

Note. As you have seen in linear algebra, if vector space has one basis with n vectors in it, then every basis for that space has n vectors. This number n is the *dimension* of the space.

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