Section 1.4. Normed Spaces

Note. In this section we consider analytic and topological properties of normed spaces.

Definition 1.4.1. A real valued function $\|\cdot\|$ on a vector space E is a *norm* if

(a) $\|x\| = 0$ if and only if $x = 0$,

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in E$ and for all $\lambda \in F$, and

(c) $\|x_y\| \le \|x\| + \|y\|$ for all $x, y \in E$.

Example/Definition. $\mathbb{R}^n$ is a normed vector space with

$$\|x\| = \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The is the *Euclidean norm.*

Example. $\mathbb{R}^n$ is a normed vector space with

$$\|x\| = \|(x_1, x_2, \ldots, x_n)\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Example. ℓ^p is a normed vector space with

$$\|z\| = \|\{z_n\}\| = \left(\sum_{n=1}^{\infty} |z_n|^p\right)^{1/p}.$$

Note. In a vector space, a norm $\|\cdot\|$ yields a metric $d(\cdot,\cdot)$ through the definition $d(x,y) := \|x-y\|$. A metric $d(\cdot,\cdot)$ yields a norm $\|\cdot\|$ through the definition $\|x\| = d(x,0)$.

Note. Just as absolute value can be used to define limits and convergence, a norm can be used for these definitions in a normed vector space.

Definition 1.4.3. Let $(E, \|\cdot\|)$ be a normed vector space. The sequence $\{x_n\}$ of elements of E *converges* to $x \in E$ if for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $n \geq M$ we have $\|x_n - x\| < \varepsilon$. This is denoted $\lim x_n = x$ or $x_n \to x$.

Note. The following are properties of limits (see 1.18):

1. A convergent sequence has a unique limit.

2. If $x_n \to x$ and $\lambda_n \to \lambda$ then $\lambda_n x_n \to \lambda x$.

3. If $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.

Definition. Consider the space $\mathcal{C}(\Omega)$ of all continuous functions on a closed and bounded set $\Omega \subset \mathbb{R}^n$ (i.e., Ω is compact). A sequence $\{f_n\} \subset \mathcal{C}(\Omega)$ *converges uniformly* to f if for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $x \in \Omega$ and for all $n \geq M$ we have $\|f(x) - f_n(x)\| < \varepsilon$.

Definition 1.4.4. Two norms on a vector space are *equivalent* if they define the same convergence (i.e., a sequence converges to 0 under one norm if and only if it converges to 0 under the other norm).

Theorem 1.4.1. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms in a vector space E. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there exist positive α and β such that $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$ for all $x \in E$.

Note. In Example 1.4.5, we see that "balls" in a normed vector space may not be round, depending on the norm.

Definition 1.4.6. A subset S of a normed space E is *open* if for all $x \in s$ there exists $\varepsilon > 0$ such that $B(x,\varepsilon) = \{y \in E \mid \|y - x\| < \varepsilon\} \subset S$. A subset S is *closed* if its complement $E \setminus S$ is open.

Note. The following theorem gives some properties of an open set in real analysis and also in the definition of a topology.

Theorem 1.4.2.

(a) The union of any number of open sets is open.

(b) The intersection of a finite number of open sets is open.

(c) The union of a finite number of closed sets is closed.

(d) The intersection of any number of closed sets is closed.

(e) The empty set and the whole space are both open and closed.

Note. A familiar result from real analysis also holds in normed vector space, as follows.

Theorem 1.4.3. A subset S of a normed space E is closed if and only if for all convergent sequences of elements of S has its limit in S.

Definition 1.4.7. Let S be a subset of a normed vector space E. The *closure* of S, denoted $\text{cl}(S)$, is the intersection of all closed sets containing S.

Definition 1.4.8. A subset S of a nomred vector space E is *dense in E* if $\text{cl}(S) = E$.

Example 1.4.8. A classic example from approximation theory is the fact that the set of polynomials on $[a, b]$ is dense in $\mathcal{C}([a, b])$ (the vector space of all functions continuous on $[a, b]$).

Theorem 1.4.5. Let S be a subset of a normed vector space E. The following are equivalent:

(a) S is dense in E.

(b) For all $x \in E$ there exists $\{x_n\} \subset S$ such that $x_n \to n$.

(c) Every nonempty open subset of E contains an element of S.

Definition 1.4.9. A subset S of a normed vector space E is *compact* if every sequence $\{x_n\} \subset S$ contains a convergent subsequence whose limit is in S.

Example 1.4.9. In $\mathbb{R}^n$ and $\mathbb{C}^n$, sets are compact if they are closed and bounded (this is the Heine-Borel Theorem).

Theorem 1.4.6. Compact sets are closed and bounded (in general).

Note. In an infinite dimensional space such as ℓ^2, there exists closed and bounded sets which are not compact!

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