Section 1.5. Banach Spaces

Note. In this section we define completeness, Banach space, and give some examples of Banach spaces.

Definition 1.5.1. A sequence $\{x_n\}$ in a normed vector space is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > M$.

Theorem 1.5.1. The following are equivalent:

(a) $\{x_n\}$ is a Cauchy sequence.

(b) $\|x_{p_n} - x_{q_n}\| \to 0$ as $n \to \infty$ for all pairs of strictly increasing sequences of positive integers $\{p_n\}$ and $\{q_n\}$.

(c) $\|x_{p_{n+1}} - x_{p_n}\| \to 0$ as $n \to \infty$ for every strictly increasing sequence of positive integers $\{p_n\}$.

Example. $\mathbb{R}^n$ is a normed vector space with

$$\|x\| = \|(x_1, x_2, \ldots, x_n)\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Corollary 1.5.A. Every convergent sequence is Cauchy.

Note. The converse of this corollary is not true. Consider the space $\mathcal{P}([0,1])$ of all polynomials on $[0,1]$ with norm $\|P\| = \max_{x\in[0,1]} |P(x)|$. Consider the sequence

$$\{P_n(x)\} = \left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right\}.$$

This sequence converges in $\mathcal{C}([0,1])$ under the same norm (to $f(x) = e^x$) and so is Cauchy. However, $\{P_n(x)\}$ does not converge in $\mathcal{P}([0,1])$. Therefore, in general, Cauchy sequences are not convergent.

Lemma 1.5.1. If $\{x_n\}$ is a Cauchy sequence in a normed vector space, then the sequence of real numbers $\{\|x_n\|\}$ converges.

Note. See page 19 for an easy proof of Lemma 1.5.1. An implication is that a Cauchy sequence is bounded.

Definition 1.5.2. A normed vector space is *complete* if every Cauchy sequence of elements of E converges to an element of E. A complete normed vector space is a *Banach space.*

Theorem 1.5.A/Example 1.5.1. The space ℓ^2 is a Banach space.

Definition 1.5.3. A series $\sum_{n=1}^{\infty} x_n$ *converges* in a normed space E if the sequence of partial sums converges in E, i.e. there exists $x \in E$ such that

$$\left\| \left(\sum_{k=1}^{n} x_k \right) - x \right\| \to 0 \text{ as } n \to \infty.$$

This is denoted $\sum_{n=1}^{\infty} x_n = x$. If $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then the series is *absolutely convergent*.

Note. A result seen in calculus is the following.

Theorem 1.5.2. A normed space is complete if and only if every absolutely convergent series converges.

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