Section 1.6. Linear Mappings

Note. In this section we define linear mappings between vector spaces and make a vector space out of the linear mappings themselves.

Definition. Let E_1 and E_2 be vector spaces. Let L be a mapping from E_1 to E_2. We take the definitions of *image*, *inverse image*, *domain*, and *range* as given on page 23 of the text. The *null space* of L, denoted $\mathcal{N}(L)$, is the set of all vectors $x \in E_1$ such that $L(x) = 0 \in E_2$.

Definition 1.6.1. A mapping $L : E_1 \to E_2$ is a *linear mapping* if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

for all $x, y \in E_1$ and for all scalars α, β.

Note. For any y in the span of the domain of L, we can use the linearity of L to define $L(y)$. Therefore, the domain of L is a vector space.

Example. $L = d/dx$ is a linear mapping from $C^2([a, b])$ to $C^1([a, b])$.

Definition 1.6.2. Let E_1 and E_2 be normed vector spaces and let L be a mapping from E_1 to E_2. If for all sequences $\{x_n\} \subset E_1$ convergent to $x_0 \in E_1$, the sequence $\{L(x_n)\}$ converges to $L(x_0)$, then mapping L is *continuous* at x_0.

Theorem 1.6.1. Let $f : E_1 \to E_2$. The following are equivalent:

(a) f is continuous (on E_1).

(b) For all open $U \subset E_2$, $f^{-1}(U)$ is open in E_1.

(c) For all closed $F \subset E_2$, $f^{-1}(F)$ is closed in E_1

Note. We address continuity of linear mappings in the following two theorems.

Theorem 1.6.2. A linear mapping $L : E_1 \to E_2$ is continuous if and only if it is continuous at a point.

Definition 1.6.3. A linear mapping $L : E_1 \to E_2$ is *bounded* if there exists $K \in \mathbb{R}$ such that $\|L(x)\| \leq K\|x\|$ for all $x \in E_1$.

Theorem 1.6.3. A linear mapping is continuous if and only if it is bounded.

Note. It turns out that the set of all continuous linear mappings of a vector space into its scalar field is itself a vector space! We need some notation and terminology.

Definition. Let L_1 and L_2 be linear mappings from E_1 to E_2. Define the *sum scalar multiplication* as

$$(L_1 + L_2)(x) = L_1(x) + L_2(x) \text{ and } (\lambda L)(x) = \lambda L(x),$$

respectively. (Notice that scalar λ must come from the scalar field of E_2.) Denote by $\mathcal{B}(E_1, E_2)$ the set of all bounded linear mappings from E_1 to E_2. For $L \in \mathcal{B}(E_1, E_2)$ define the *norm*

$$\|L\| = \sup_{\|x\|=1} \|L(x)\|.$$

Theorem 1.6.4. If E_1 and E_2 are normed spaces then $\mathcal{B}(E_1, E_2)$ is a normed space with norm as given above.

Note/Definition. Convergence in $\mathcal{B}(E_1, E_2)$ with respect to the above norm is *uniform convergence.*

Definition. A sequence $\{L_n\} \subset \mathcal{B}(E_1, E_2)$ *converges strongly* to $L \in \mathcal{B}(E_1, E_2)$ if for all $x \in E_1$ we have $\|L_n(x) - L(x)\| \to 0$ as $n \to \infty$. (Notice that $\|L_n(x) - L(x)\| \le \|L_n - L\| \, \|x\|$ and so uniform convergence implies strong convergence; the converse does not generally hold.)

Theorem 1.6.5. If E_1 is a normed space and E_2 is a Banach space, then $\mathcal{B}(E_1, E_2)$ is a Banach space.

Theorem 1.6.6. Let f be a continuous linear mapping from a subspace of a normed space E_1 into a Banach space E_2. Then f has a unique extension to a continuous mapping defined on the closure of $\mathcal{D}(f)$.

Definition. Let E be a normed vector space with scalar field F. Elements of $\mathcal{B}(E, F)$ are called *functionals*. The space $\mathcal{B}(E, F)$ is called the *dual space* of E.

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