Section 3.11. Linear Functionals and the Riesz Representation Theorem

Note. In this section we will see that every bounded linear functional f on a Hilbert space is of the form $f(x) = (x, x_0)$ for all $x \in H$ and for some given $x_0 \in H$.

Example 3.11.2. Let $H = L^2([a,b])$ and let $t_0 \in [a,b]$. Define a functional f on H as $f(x) = x(t_0)$ (that is, $f(x)$ is the value of function x at point t_0). Then f is linear:

$$f(x+y) = (x+y)(t_0) = x(t_0) + y(t_0) = f(x) + f(y).$$

Define the sequence $\{x_n\} \subset H$ as

$$x_n(t) = \begin{cases} n \text{ if } t = t_0 \\ 0 \text{ if } t \neq t_0. \end{cases}$$

Then f is not bounded on $\{x_n\}$ since $\{f(x_n)\} = \{n\} \to \infty$. Also, $\lim_{n\to\infty} x_n = 0$ and

$$f\left(\lim_{n\to\infty} x_n\right) = f(0) = 0 \neq \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = \infty.$$

So this is an example of a linear functional which is NOT continuous.

Theorem 3.11.1. Riesz Representation Theorem.

Let f be a bounded linear functional on a Hilbert space H. There exists exactly one $x_0 \in H$ such that $f(x) = (x, x_0)$ for all $x \in H$. Also $\|f\| = \|x_0\|$.

Note. We have seen that the collection of all bounded linear functional on a Hilbert space are a Banach space (Theorem 1.6.5), called the dual space. We now see that the dual space of a Hilbert space H is isomorphic to H itself.

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