Section 3.12. Separable Hilbert Spaces

Note. In this section we define a Hilbert space isomorphism and, in a sense, classify infinite dimensional Hilbert spaces (in the sense that the space must be separable).

Definition 3.12.1. A Hilbert space is *separable* if it contains a complete orthonormal sequence.

Note. So finite dimensional Hilbert spaces are separable. A Hilbert space H is separable if and only if it has a Schauder basis. See my online notes on "Vector Spaces and Hilbert Spaces: Norms, Completeness, Inner Products, and Hilbert Spaces" at `http://faculty.etsu.edu/gardnerr/talks/Butt-Head2.pdf`.

Note. Example 3.12.3 gives an example of a non-separable Hilbert space.

Theorem 3.12.1. Every separable Hilbert space contains a countable dense subset.

Theorem 3.12.2. Every set of mutually orthogonal vectors in a separable Hilbert space is countable.

Definition 3.12.2. A Hilbert space H_1 is *isomorphic* to a Hilbert space H_2 if there exists a one to one linear mapping T from H_1 onto H_2 such that $(T(x), T(y)) = (x, y)$ for all $x, y \in H_1$.

Theorem 3.12.3. Fundamental Theorem of Finite and Infinite Dimensional Vector Spaces.
Let H be a separable Hilbert space over scalar field $\mathbb{C}$. Then:

(a) if H is finite dimensional, then H is isomorphic to $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and

(b) if H is infinite dimensional then H is isomorphic to ℓ^2.

Note. The title "Fundamental Theorem of Finite and Infinite Dimensional Vector Spaces" is due to me and has appeared in print in D. Hong, J. Wang, and R. Gardner's *Real Analysis with an Introduction to Wavelets*, Academic Press/Elsevier Press (2005); see Chapter 5, "Vector Spaces, Hilbert Spaces, and the L^2 Space" or my online notes at `http://faculty.etsu.edu/gardnerr/Func/notes-HWG.htm`.

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