Section 3.4. Norm in an Inner Product Space

Note. In this section we explore some geometric ideas in an inner product space.

Definition. In an inner product space, define $\|x\| = \sqrt{(x, x)}$.

Note. With $\| \cdot \|$ defined as above, $\|x\| = 0$ if and only if $x = 0$ and $\|\lambda x\| = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda\overline{\lambda}(x, x)} = |\lambda|\|x\|$ where $\lambda \in \mathbb{C}$ is a scalar. If we establish the Triangle Inequality, then we have shown that $\| \cdot \|$ is a norm.

Theorem 3.4.1. Schwarz's Inequality.
For any x and y in an inner product space $|(x, y)| \leq \|x\| \, \|y\|$. Equality holds if and only if x and y are linearly dependent.

Note. As in Linear Algebra, we use Schwarz's Inequality to prove the Triangle Inequality for norm.

Corollary 3.4.1. For any two elements x and y of an inner product space we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

Note. The following is a property of the norm induced by an inner product. If you encounter the normed spaces ℓ^p and $L^p([a, b])$ where $p \geq 1$ (i.e., the classical Banach spaces) then you can show that they are not inner product spaces, except in the case $p = 2$, by showing that the norm for $p \neq 2$ violates the following.

Theorem 3.4.2. Parallelogram Law.

For any two elements x and y of an inner product space

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Note. The main use of an inner product is to establish orthogonality. This will be especially useful when considering bases for an inner product space.

Definition 3.4.2. Two vectors x and y in an inner product space are *orthogonal* if $(x, y) = 0$.

Theorem 3.4.3. Pythagorean Theorem.

If x and y are orthogonal then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Note. The fact that the Pythagorean Theorem holds in an inner product space implies that the geometry of these spaces is Euclidean. Recall that the Pythagorean Theorem is equivalent to the Parallel Postulate in classical plane geometry.

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