Section 3.5. Hilbert Spaces—Definition and Examples

Note. In this section we define and give a few examples of Hilbert spaces.

Definition. A complete inner product space is a *Hilbert space*.

Note. Schematically, we have

$$\text{Hilbert spaces} \subset \text{Banach space} \subset \text{vector space.}$$

Note. All finite dimensional spaces over $\mathbb{R}$ and $\mathbb{C}$ are complete (see Exercise 3.15).

Example 3.5.3. ℓ^2 is a Hilbert space, as proved in Example 1.5.1.

Example 3.3.4/3.5.4. Consider the space of sequences $\{x_n\}$ of complex numbers such that only a finite number of terms are nonzero. The sequence $x_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$ is Cauchy, but does not converge in this space (it does converge in ℓ^2, though). This is an "incomplete" inner product space.

Theorem/Example 3.5.6. $L^2([a, b])$ (the linear space of all square [Lebesgue] integrable functions on $[a, b]$) is a Hilbert space.

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