Section 3.6. Strong and Weak Convergence

Note. In this section we consider weak convergence with respect to the inner product.

Definition 3.6.1. A sequence $\{x_n\}$ of vectors in an inner product space E is *strongly convergent* to $x \in E$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

Note. Strong convergence is just convergence under the norm of the inner product space.

Definition 3.6.2. A sequence $\{x_n\}$ is *weakly convergent* to $x \in E$ if for all $y \in E$

$$(x_n, y) \to (x, y) \text{ as } n \to \infty.$$

We denote this as $x_n \xrightarrow{w} x$.

Theorem 3.6.1. A strongly convergent sequence is weakly convergent (to the same limit).

Note. The converse of Theorem 3.6.1 does not hold. Consider

$$x_n = (0, 0, \ldots, 0, 1, 0, \ldots) \in \ell^2 \text{ (where the } n\text{th entry is 1).}$$

For any $y = (y_1, y_2, \ldots) \in \ell^2$, there exists $m \in \mathbb{N}$ such that for all $\varepsilon > 0$ we have $|y_i| < \varepsilon$ for $i > m$. Therefore for any $y \in \ell^2$, $(x_n, y) \to 0 = (x_0, y)$ as $n \to \infty$ (i.e., $x_n \xrightarrow{x} x_0 = (0, 0, \ldots))$, but $x_n \not\to x_0$.

Theorem 3.6.2. If $x_n \rightarrow_w x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Note. For any given $y \in E$, $(\cdot, y) : E \to \mathbb{C}$ is a linear functional. Theorem 3.6.1 implies that this functional is continuous.

Theorem 3.6.3. Weakly convergent sequences are bounded.

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