Section 3.8. Properties of Orthonormal Systems

Note. In this section we consider more orthonormal systems, with special attention to their behavior as a basis.

Theorem 3.8.1. Pythagorean Formula.

If $x_1, x_2, \ldots, x_n$ are orthogonal then

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2.$$

Proof. The proof follows by induction or directly from the definition of $\|\cdot\|^2$ in terms of the inner product. $\square$

Theorem 3.8.2. Bessel Equality and Inequality.

Let $x_1, x_2, \ldots, x_n$ be an orthonormal set of vectors in an inner product space E. Then for all $x \in E$,

$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

and $\sum_{k=1}^{n} |(x, x_k)|^2 \leq \|x\|^2$.

Note/Definition. The proof of Theorem 3.8.4 shows that $\left\|x - \sum_{k=1}^{n} \alpha_k x_k\right\|^2$ is minimum when $\alpha_k = (x, x_k)$. Therefore the *best approximation* of x with a linear combination of $\{x_1, x_2, \ldots, x_n\}$ is $\sum_{k=1}^{n} (x, x_k) x_k$.

Definition. Let $\{x_1, x_2, \ldots\}$ be an orthonormal sequence in an inner product space E. Then for $x \in E$, $\sum_{k=1}^{\infty}(x, x_k)x_k$ is the *generalized Fourier series* for x and (x, x_k) are the *generalized Fourier coefficients*.

Theorem 3.8.3. Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H and let $\{\alpha_n\} \subset \mathbb{C}$. The series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and then

$$\left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\|^2 = \sqrt{\sum_{n=1}^{\infty} |\alpha_n|^2}.$$

Theorem 3.8.3. Let $\{x_n\}$ be an orthonormal sequence is a Hilbert space and let $\{\alpha_n\} \subset \mathbb{C}$ be a sequence. Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\{\alpha_n\} \in \ell^2$. Then

$$\left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\| = \sqrt{\sum_{n=1} \infty |\alpha_n|^2}.$$

Note. For a given orthonormal sequence and vector x, the series $\sum_{n=1}^{\infty}(x, x_n)x_n$ converges, but need not converge to x, as shown in the following example.

Example 3.8.1. Let $H = L^2([-\pi, \pi])$ and let $x_n = \sqrt{\pi} \sin nt$. Then $\{x_n\}$ is an orthonormal set (BUT NOT A BASIS) in H. Consider $x(t) = \cos t$ and we have

$$\sum_{n=1}^{\infty}(x, x_n)x_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{\pi} \cos t \sin nt \, dt\right) \sin nt = 0 \neq \cos t.$$

Note. If $\sum_{n=1}^{\infty} |(x, x_n)|^2 < \infty$ then $\lim_{n\to\infty}(x, x_n) = 0$ and so all orthonormal sequences weakly converge to 0.

Definition 3.8.1. An orthonormal sequence $\{x_n\}$ in a Hilbert space H is *complete* if for all $x \in H$

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n.$$

Theorem 3.8.4. An orthonormal sequence $\{x_n\}$ in a Hilbert space H is complete if and only if the condition $(x, x_n) = 0$ for all $n \in \mathbb{N}$ implies $x = 0$.

Theorem 3.8.5. Parseval's Formula.

An orthonormal sequence $\{x_n\}$ in a Hilbert space H is complete if and only if

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2$$

for all $x \in H$.

Example 3.8.4. In $L^2([0, \pi])$ the sequences

$$\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{ \sqrt{\frac{2}{\pi}} \cos nx \;\middle|\; n \in \mathbb{N} \right\} \text{ and } \left\{ \sqrt{\frac{2}{\pi}} \sin nx \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\}$$

are each complete orthonormal system in $L^2([0, \pi])$. Notice that this means that for any $f \in L^2([0, \pi])$ that $|f(x) - \sum_{n=0}^{\infty} \alpha_n x_n| \to 0$ as $n \to \infty$ where x_n is chosen from above and $\alpha_n = \int_0^{\pi} f \overline{x}_n$. This does not mean that $\sum_{n=0}^{\infty} \alpha_n x_n \to f$ POINTWISE, but instead that f is the limit under the L^2 norm!