Section 4.10. Spectral Decomposition

Note. In this section we explore a generalization of the idea of eigenvalues of a matrix and consider the spectrum of a self adjoint compact operators.

Note. In finite dimensions, the eigenvectors of a self adjoint operator form an orthogonal basis for the space. A similar result holds for infinite dimensional Hilbert spaces, as follows.

Theorem 4.10.1. Hilbert Schmidt Theorem.
For every self adjoint compact operator A on an infinite dimensional Hilbert space H, there exists an orthonormal system of eigenvectors $\{u_n\}$ corresponding to eigenvalues $\{\lambda_n\}$ such that for all $x \in H$ there is a unique representation $x = \sum_{n=1}^{\infty} \alpha_n u_n + v$ where $\alpha_n \in \mathbb{C}$ and v is such that $Av = 0$.

Note. The Hilbert-Schmidt Theorem says that a Hilbert space is the direct sum (maybe infinite) of the nullspace and eigenspaces of self adjoint compact operator A.

Theorem 4.10.2. Spectral Theorem for Self Adjoint Compact Operators.
Let A be a self adjoint compact operator on an infinite dimensional Hilbert space H. Then there exists in H a complete orthonormal system $\{v_n\}$ consisting of eigenvectors of A. Moreover, for all $x \in H$ we have $Ax = \sum_{n=1}^{\infty} \lambda_n (x, v_n) v_n$, where λ_n is the eigenvalue corresponding to v_n.

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