Section 4.11. The Fourier Transform

Note. In this section we define the Fourier transform, initially on $L^1(\mathbb{R})$ and later on $L^2(\mathbb{R})$. We discuss properties in both settings.

Note. Let $f \in L^1(\mathbb{R})$ (that is, let f be measurable and integrable on $\mathbb{R}$). Then for fixed $k \in \mathbb{R}$,

$$\left|\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\right| \leq \int_{-\infty}^{\infty} |e^{-ikx} f(x)|\,dx = \int_{-\infty}^{\infty} |f(x)|\,dx < \infty,$$

where the first inequality holds by, say, Proposition IV.1.17(b) of IV.1. Riemann-Stieltjes Integrals in my online notes for Complex Analysis 1 (MATH 5510), and the equality holds since $|e^{-ikx}| = 1$. So $e^{-ikx} f(x)$ is integrable and in $L^1(\mathbb{R})$.

Definition 4.11.1. Let $f \in L^1(\mathbb{R})$. The function

$$\mathscr{F}\{f\} = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

is the *Fourier transform* of f.

Example 4.11.1(b). For $f(x) = e^{-x^2}$ we have

$$\mathscr{F}(f) = \mathscr{F}(e^{-x^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx)}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx-k^2/4_-k^2/4}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/4} \int_{-\infty}^{\infty} e^{-(x+ik/2)^2}\,dx \text{ let } u = x + ik/2, \text{ so } du = dx$$

$$\frac{1}{\sqrt{2\pi}} e^{-k^2/4} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{2\pi}} e^{-k^2/4} \sqrt{\pi} = \frac{1}{\sqrt{2}} e^{-k^2/4}$$

since $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Note. It follows directly from the definition of Fourier transform that $\mathscr{F}$ is linear, as follows.

Theorem 4.11.2. The Fourier transform of an integrable function is a continuous function.

Note. Since equation (4.11.3) is independent of k, then the proof of Theorem 4.11.2 actually shows that $\mathscr{F}(f) = \hat{f}$ is uniformly continuous on $\mathbb{R}$.

Theorem 4.11.3. If $f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = \|f_n - f\|_1 \to 0$ as $n \to \infty$ then the sequence of Fourier transforms $\{\hat{f}_n\}$ converges to $\hat{f}$ uniformly on $\mathbb{R}$.

Theorem 4.11.4. The Riemann-Lebesgue Theorem.
If $f \in L^1(\mathbb{R})$ then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Note. The space $\mathscr{C}_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ satisfying $\lim_{|x|\to\infty} f(x) = 0$ is a normed linear space with norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. Theorems 4.11.1 to 4.11.4 show that the Fourier transform is a continuous linear operator from $L^1(\mathbb{R})$ to $\mathcal{C}_0(\mathbb{R})$, $\mathscr{F} : L^1(\mathbb{R}) \to \mathscr{C}_0(\mathbb{R})$.

Note. The following result follows directly from the definition of Fourier transform.

Theorem 4.11.5. Let $f \in L^1(\mathbb{R})$. Then

(a) $\mathscr{F}\{\hat{f}(x)\} = \overline{\mathscr{F}\{f(-x)\}}$ (conjugate),

(b) $\mathscr{F}\{f(x-y)\} = \mathscr{F}\{f(x)\}e^{-iky}$ (shifting),

(c) $\mathscr{F}\{f(\alpha x)\} = (1/\alpha)\mathscr{F}\{f(x/\alpha)\}$, $\alpha > 0$ (scaling), and

(d) $\mathscr{F}\{e^{i\alpha x} f(x)\} = \mathscr{F}\{f(x-\alpha)\}$ (translation).

Theorem 4.11.6. If f is a continuous piecewise differentiable function, $f, f' \in L^1(\mathbb{R})$, and $\lim_{|x|\to\infty} f(x) = 0$ then $\mathscr{F}\{f'\} = ik\mathscr{F}\{f\}$.

Note. By induction, we have the following corollary.

Corollary 4.11.1. If f is a continuous piecewise n-times differentiable function, where $f, f', f'', \dots, f^{(n)} \in L^1(\mathbb{R})$, and $lim_{|x|\to\infty} f^{(k)}(x) = 0$ for $k = 0, 1, \dots, n-1$ then $\mathscr{F}\{f^{(n)}\} = i^n k^n \mathscr{F}\{f\}$.

Definition. The *convolution* of $f, g \in L^1(\mathbb{R})$ is (now) defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u)g(u)\,du.$$

Note. The above definition varies from the definition of convolution gives in "Section 2.15. Convolution" in that a factor of $1/\sqrt{2\pi}$ is introduced. This allows us to state a clear result concerning the Fourier transformation of a convolution as follows.

Theorem 4.11.7. Convolution Theorem.
Let $f, g \in L^1(\mathbb{R})$. Then $\mathscr{F}\{f * g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$.

Note. We want to extend the Fourier transform to $L^2(\mathbb{R})$. First, we need a preliminary result. Recall that the norm on $L^2(\mathbb{R})$ is $\|f\|_2 = \left\{\int_{-\infty}^{\infty} |f(x)|^2\, dx\right\}^{1/2}$.

Theorem 4.11.8. Let f be a continuous function on $\mathbb{R}$ vanishing outside a bounded interval. Then $\hat{f} =\in L^2(\mathbb{R})$ and $\|\hat{f}\|_2 = \|f\|_2$.

Note. The space of all continuous functions on $\mathbb{R}$ which vanish outside a bounded interval is dense in $L^2(\mathbb{R})$ (see Theorem 7.12 of my online Real Analysis [MATH 5210/5220] notes on 7.4. Approximation and Separability). Theorem 4.11.8 shows that $\mathscr{F}$ maps this space of functions to $L^2(\mathbb{R})$ in a distance preserving way (such a map is called an *isometry*). So with $\delta = \varepsilon$, if $\|f_1 - f_2\|_2 < \delta$ then

$$\|\mathscr{F}\{f_1\} - \mathscr{F}\{f_2\}\|_2 = \|\hat{f}_1 - \hat{f}_2\| < \varepsilon.$$

That is, $\mathscr{F}$ is a continuous mapping from the space to $L^2(\mathbb{R})$. Since $\mathscr{F}$ is linear by Theorem 4.11.1, then there is a unique extension of $\mathscr{F}$ from this space (which is dense in $L^2(\mathbb{R})$) to $L^2(\mathbb{R})$. This allows us to extend $\mathscr{F}$ to $L^2(\mathbb{R})$, as follows.

Definition 4.11.2. Let $f \in L^2(\mathbb{R})$ and let $\{\varphi_n\}$ be a sequence of continuous functions with compact support which converges to f in $L^2(\mathbb{R})$; that is, $\|f-\varphi_n\|_2 \to 0$ (this can be done since the continuous function vanishing outside a bounded interval are dense in $L^2(\mathbb{R})$). The *Fourier transform* of f is $\hat{f} = \lim_{n\to\infty} \hat{\varphi}_n$, where the limit is with respect to the norm in $L^2(\mathbb{R})$.

Note. In Definition 4.11.1 (the Fourier transformation of an $L^1(\mathbb{R})$ function), a pointwise definition of $\mathscr{F}\{f\}$ is given (that is, $\hat{f}(k)$ is defined for $k \in \mathbb{R}$). In Definition 4.11.2 (the Fourier transform of an $L^2(\mathbb{R})$ function) we use convergence with respect to the $L^2(\mathbb{R})$ norm. So $\hat{f}$ is not determined at individual points, but only "up to" sets of measure zero. So for f
$inL^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we may not have $\hat{f}(k) = \hat{f}$ (where we use Definition 4.11.1 to find $f(k)$ and Definition 4.11.2 to find $\hat{f}$) but we will have that $\|\hat{f}(k) = \hat{f}\|_2 = 0$; that is, $\hat{f}(k)$ is in the same equivalence class in $L^2(\mathbb{R})$ as is $\hat{f}$ so that as elements of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we would still write "$\hat{f}(k) = \hat{f}$."

Note. The next result shows that Theorem 4.11.8 also holds for the Fourier transform on $L^2(\mathbb{R})$.

Theorem 4.11.9. Parseval's Relation.
If $f \in L^2(\mathbb{R})$ then $\|\hat{f}\|_2 = \|f\|_2$.

Note. Notice that Parseval's Relation implies that if $f \in L^2(\mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R})$. The following allows us to express the Fourier transform on $L^2(\mathbb{R})$ as a limit of definite integrals.

Theorem 4.11.10. Let $f \in L^2(\mathbb{R})$. Then

$$\hat{f}(k) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Note. Next, we want to define the inverse Fourier transformation on $L^2(\mathbb{R})$. We need a theorem and a "technical lemma" first.

Theorem 4.11.11. Weak Parseval's Relation.
If $f, g \in L^2(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x) g(x)\, dx.$$

Lemma 4.11.1. Let $f \in L^2(\mathbb{R})$ and let $g = \overline{\hat{f}}$. Then $f = \overline{\hat{g}}$.

Note. We can now define the inverse Fourier transform on $L^2(\mathbb{R})$.

Theorem 4.11.12. Inversion of Fourier Transform on $L^2(\mathbb{R})$.
Let $f \in L^2(\mathbb{R})$. Then

$$f(x) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(k)\, dk$$

where the convergence is with respect to the norm in $L^2(\mathbb{R})$.

Note. Since elements of $L^2(\mathbb{R})$ are equivalence classes and we have defined the Fourier transform for $f \in L^1(\mathbb{R})$ pointwise (see Definition 4.11.1), then for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have the following.

Corollary 4.11.2. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the equality

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk$$

holds almost everywhere for $x \in \mathbb{R}$.

Definition For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, define the *inverse Fourier transform*

$$\mathscr{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk$$

where $\hat{f}(x) = \mathscr{F}\{f(x)\} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$.

Theorem 4.11.13. **General Persaval's Relation.**
If $f, g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}}\, dk.$$

Note. For $f \in L^2(\mathbb{R})$, the previous theorems of this section imply the following.

Theorem 4.11.14. Plancherel's Theorem.

For every $f \in L^2(\mathbb{R})$ there exists $\hat{f} \in L^2(\mathbb{R})$ such that:

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$.

(b) $\left\| \hat{f}(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\,dx \right\|_2 \to 0$ and $\left\| f(k) - \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{ikx} \hat{f}(x)\,dx \right\|_2 \to 0$ as $n \to \infty$.

(c) $\|f\|_2 = \|\hat{f}\|_2$.

(d) The mapping $f \mapsto \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Note. Recall that the adjoint T^* of a bounded operator T on a Hilbert space H is defined by the relation $(Tx, y) = x, T^*y)$ for all $x, y \in H$ (see Definition 4.4.1). A bounded operator T is unitary if $T^*T = TT^* = \mathcal{I}$; that is, $(T^*Tx, x) = (Tx, Tx) = (x, x)$ for all x (see Definition 4.5.4 and Theorem 4.5.9).

Theorem 4.11.15. The Fourier transform is an unitary operator on $L^2(\mathbb{R})$.

Note. We can also define a Fourier transform on $L^1(\mathbb{R}^n)$ as

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ikx} f(x)\,dx$$

where $k = (k_1, k_2, \ldots, k_n)$ and $x = (x_1, x_2, \ldots, x_n)$ are in $\mathbb{R}^n$ and $k\dot{x} = k_1x_1 + k_2x_2 + \cdots + k_nx_n$. The extension to $L^2(\mathbb{R}^n)$ is possible and must of the theory of this section extends to $L^2(\mathbb{R}^n)$, such as the Inversion Theorem and the Plancherel Theorem.

Note. In Section 5.11 (included as a supplement to these notes) we present applications of the Fourier transform to ordinary differential equations and integral equations.

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