Section 4.5. Invertible, Normal, Isometric, and Unitary Operators

Note. In this section we explore several types of linear operators.

Definition 4.5.1. Let A be an operator defined on a vector subspace of vector space E. An operator B defined on the range of A, $\mathcal{R}(A)$, is the *inverse* of A if $ABx = x$ for all $x \in \mathcal{R}(A)$ and $BAx = x$ for all $x \in \mathcal{D}(A)$ (the domain or A). We denote $B = A^{-1}$.

Theorem 4.5.1. The inverse of a linear operator is linear.

Note. The following property of linear operators is similar to the result concerning matrices that states that square matrix A is invertible if and only if the nullspace of A is $\{0\}$.

Theorem 4.5.2. Linear operator A is invertible if and only if $Ax = 0$ implies $x = 0$.

Theorem 4.5.3. If an operator A is invertible and vectors $x_1, x_2, \ldots, x_n$ are linearly independent then $Ax_1, Ax_2, \ldots, Ax_n$ are linearly independent.

Note. The proof of Theorem 4.5.3 follows from Theorem 4.5.2.

Note. Example 4.5.2 shows that the inverse of a bounded linear operator may not be bounded. The next theorem addresses this observation.

Theorem 4.5.5. Let A be a bounded linear operator with range a Hilbert space H. If A has a bounded inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Corollary 4.5.1. If a bounded self adjoint operator A has abounded inverse A^{-1}, then A^{-1} is self adjoint.

Definition 4.5.2. A bounded operator T is a *normal operator* if it commutes with its adjoint: $TT^* = T^*T$.

Theorem 4.5.8. Let T be a bounded operator on a Hilbert space H and let A and B be self adjoint operators on H such that $T = A + iB$. Then T is normal if and only if A and B commute.

Note. Our analogy of normal operators and complex numbers is complete. The adjoint acts as conjugation and so from Theorem 4.5.8 we have the analogy of modulus: $TT^* = A^2 + B^2$.

Definition 4.5.3. A bounded operator T on a Hilbert space is an *isometric operator* if $\|Tx\| = \|x\|$ for all $x \in H$.

Theorem 4.5.9. A bounded linear operator T on a Hilbert space H is isometric if and only if $T^*T = \mathcal{I}$ on H.

Note. Not only does an isometric operator preserve "lengths," but preserves inner products:

$$(Tx, Ty) = (x, T^*Ty) = (x, y).$$

Definition 4.5.4. A bounded linear operator T on a Hilbert space H is a *unitary operator* if $T^*T = TT^* = \mathcal{I}$ on H.

Note. Trivially, every unitary operator is normal (see Theorem 4.5.10).

Theorem 4.5.11/4.5.12. A linear operator T is unitary if and only if it is invertible and $T^{-1} = T^*$. If T is unitary then T^{-1} and T^* are unitary.

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