Section 4.9. Eigenvalues and Eigenvectors

Note. In this section we consider eigenvalues and eigenvectors for linear operators and define the spectrum of an operator. We will see some parallel behaviors between results for matrices and operators.

Definition 4.9.1. A (complex) number λ is an *eigenvalue* of linear operator A if there exists vector $u \neq 0$ such that $Au = \lambda u$. Such a vector u is an *eigenvector* of A.

Definition 4.9.2. Let A be an operator on a normed space. The operator $A_\lambda = (A - \lambda\mathcal{I})^{-1}$ is the *resolvent* of A. The values of λ for which A_λ is defined on the whole space and is bounded are the *regular points* of A. The set of all λ which are not regular in the *spectrum* of A. The eigenvalues of A (a subset of the spectrum) is the *point spectrum* of A. The remainder of the spectrum (for which A_λ exists by is unbounded) is the continuous spectrum.

Example 4.9.2. Let $= \mathcal{C}([a, b])$ and $u \in E$. Define $A : E \to E$ as $(Ax)(t) = u(t)x(t)$. The resolvent of A is

$$(A - \lambda\mathcal{I})^{-1} = \frac{x(t)}{u(t) - \lambda}.$$

Now A_λ is defined and bounded on all of E if λ is such that $u(t) \neq \lambda$ for all $t \in [a, b]$. That is, if λ is not in the range of u then λ is a regular point of A. The points in the range of u make up the spectrum of A. Only if $u(t)$ is a constant function, $u(t) = \lambda$, does A have eigenvalues.

Theorem 4.9.1. The collection of all eigenvectors corresponding to one particular eigenvalue of an operator is a vector space.

Definition 4.9.3. The vector space of the previous theorem is the *eigenspace* of the eigenvalue λ. The dimension of the eigenspace of λ is the *multiplicity* of λ. An eigenvalue of multiplicity one is *simple* (or *non-degenerate*). An eigenvalue of multiplicity greater than one is *multiple* (or *degenerate*) and the number of linearly independent eigenvectors is he *degree of degeneracy*.

Theorem 4.9.2. Let T be an invertible linear operator on E and let A be a linear operator on E. Then A and TAT^{-1} have the same eigenvalues.

Note. Recall that the eigenvalues of a real symmetric matrix are real. This idea is generalized as follows.

Theorem 4.9.3. All eigenvalues of a self adjoint operator on a Hilbert space are real.

Note. The following two theorems give further motivation for the names "positive operator" and "unitary operator."

Theorem 4.9.4. All eigenvalues of a positive operator are non-negative. All eigenvalues of a strictly positive operator are positive.

Theorem 4.9.5. All eigenvalues of a unitary operator on a Hilbert space are complex numbers of modulus 1.

Theorem 4.9.6. Eigenvectors corresponding to distinct eigenvalues of self adjoint or unitary operator on a Hilbert space are orthogonal.

Theorem 4.9.7. For every eigenvalue λ of a bounded operator A, we have $|\lambda| \leq \|A\|$.

Theorem 4.9.9. Let A be a self adjoint operator. Define $m = \inf_{\|x\|=1}(Ax, x)$ and $M = \sup_{\|x\|=1}(Ax, x)$. The spectrum of A lies in $[m, M]$. Also m and M are in the spectrum.

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