# Section 5.11. Applications of Fourier Transforms to Ordinary Differential Equations and Integral Equations

**Note.** We now apply the Fourier transform to both (ordinary) differential equations and integral equations. In addition to some specific applications, we give examples for ODEs (see Examples 5.11.3 and the next note) and integral equations (see examples 5.11.5 and 5.11.7).

**Note.** Consider the $n$th order (nonhomogeneous) differential equation with constant coefficients $L(y) = f(x)$ where $L$ is the $n$th order differential operator $L = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$ for $a_0, a_1, \ldots, a_n$ constants and $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Applying the Fourier transform to the ODE gives $\mathscr{F}\{L(y)\} = \mathscr{F}\{f(x)\}$ or (by Theorem 4.11.6)

$$(a_n(ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik_+a_0)\hat{y}(k) = \hat{f}(k).$$

With polynomial $\hat{p}$ defined as $\hat{p}(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ we have $\hat{p}(ik) = \hat{y}(k) = \hat{f}(k)$. With polynomial $\hat{p}$ defined as $\hat{p}(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ we have $\hat{p}(ik)\hat{y}(k_= \hat{f}(k)$. So $\hat{y}(k) = \dfrac{\hat{f}(k)}{\hat{p}(ik)} = \hat{f}(k)\hat{g}(k)$ where $\hat{g}(k) = 1/\hat{p}(ik)$. The Convolution Theorem (Theorem 4.11.7) state that

$$\mathscr{F}\{f * g\} = \mathscr{F}\{f\}\mathscr{F}\{g\} \text{ where } (f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u)g(u)\, du.$$

Since we have $\mathscr{F}\{y\} = \mathscr{F}\{f\}\mathscr{F}\{g\} = \mathscr{F}\{f * g\}$ then (since $\mathscr{F}$ is one to one by Theorem 4.11.14(d))

$$y(x) = (f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x - \xi)\, d\xi,$$

provided $g(x) = \mathscr{F}^{-1}\{\hat{g}(k)\} = \mathscr{F}^{-1}\{1/\hat{p}(ik)\}$ is known explicitly. Notice that this gives a *particular* solution to the nonhomogeneous ODE: $L(y) = f(x)$. the general solution can be found using the general solution to the homogeneous ODE with constant coefficients $L(y) = 0$; this ODE can be solved using matrix theory—see my online notes for "A Second Course in Differential Equations" class (not an official ETSU class, but some of the material is likely covered in ETSU's Introduction to Applied Math (4027/5027) on 7.5. Homogeneous Linear Systems with Constant Coefficients, 7.6. Complex Eigenvalues, 7.7. Repeated Eigenvalues, 7.8. Fundamental Matrices, and 7.9. Nonhomogeneous Linear Systems.

**Example 5.11.3.** Consider the ODE $-\dfrac{d^2u}{dx^2} + \alpha^2 u = f(x)$, where $f \in L^2(\mathbb{R})$. Applying the Fourier transform we have $\mathscr{F}\left\{-\dfrac{d^2u}{dx^2} + \alpha^2 u\right\} = \mathscr{F}\{f(x)\}$ or (by Corollary 4.11.1)

$$-(ik)^2\hat{u}(k)\alpha^2\hat{u}(k) = \hat{f}(k) \text{ or } (k^2 + \alpha^2)\hat{u}(k) = \hat{f}(k).$$

Hence $\hat{u}(k) = \dfrac{1}{k^2+\alpha^2}\hat{f}(k)$. Since (as stated on page 273 and which can be verified using the theory of residues (see my online notes for Complex Variables [MATH 4337/5337] on Chapter 6. Residues and Poles) that:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+\alpha^2}\,dk = \frac{1}{2\alpha}e^{-\alpha|x|},$$

so that $\mathscr{F}\left\{\dfrac{1}{2\alpha}e^{-\alpha|x|}\right\} = \dfrac{1}{k^2+\alpha^2}$. So we now have by the Convolution Theorem (Theorem 4.11.7),

$$\mathscr{F}\{u\} = \frac{1}{k^2+\alpha^2}\mathscr{F}\{f\} = \mathscr{F}\left\{\frac{1}{2\alpha}e^{-\alpha|x|}\right\}\mathscr{F}\{f\} = \frac{1}{2\alpha}e^{-\alpha|x|} * f.$$

Since $f$ is one to one by Theorem 4.11.14(d),

$$u(x) = \frac{1}{2\alpha}e^{-\alpha|x|} * f(x) = \frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-t|} f(t)\,dt.\square$$

**Example 5.11.4.** Consider an infinite beam on an elastic foundation under a prescribed vertical load $W(x)$. The vertical deflection $u(x)$ is described by the ODE $EIu^{(4)} + ku = W(x)$ where $EI$ is the "fluxural rigidity" and $k$ is the "foundation modulus" of the beam. We assume $W$ has compact support so that $u$, $u'$, $u''$, and $u'''$ all tens to zero as $|x| \to \infty$. First, we rewrite the ODE as $u^{(4)} + \alpha^4 u = w(x)$ where $\alpha^4 = k/EI$ and $w(x) = W(x)/EI$. Applying the Fourier transform gives $\mathscr{F}\{u^{(4)}\} + \alpha^4\mathscr{F}\{u\} = \mathscr{F}\{w\}$ and by Corollary 4.11.1

$$(ik)^4\hat{u}(k) + \alpha^4\hat{u}(k) = \hat{w}(k) \text{ or } \hat{u}(k) = \frac{1}{k^4+\alpha^4}\hat{w}(k).$$

Applying the inverse Fourier transform we get

$$u(x) = \mathscr{F}^{-1}\left\{\frac{\hat{w}(k)}{k^4+\alpha^4}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{w}(k)}{k^4+\alpha^4}e^{ikx}\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^4+\alpha^4}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ik\xi}w(\xi)\,d\xi\right)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}w(\xi)G(\xi,x)\,d\xi$$

where

$$G(\xi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ik(x-\xi)}}{k^4+\alpha^4}\,dk = \frac{1}{\pi}\int_0^{\infty}\frac{\cos(x-\xi)k\,dk}{k^4+\alpha^4}$$

where the last equality results from the Euler equation $e^{i\theta} = \cos\theta + i\sin\theta$ and the fact that $\sin\theta$ is an odd function. The last integral "can be evaluated by complex contour integration," as Debnath and Mikusinski state (see page 274) to give

$$G(\xi,x) = \frac{1}{2\alpha^3}e^{-\alpha|x-\xi|/\sqrt{2}}\sin\left(\frac{\alpha|x-\xi|}{\sqrt{2}}+\frac{\pi}{4}\right).$$

In particular, a point load of unit strength acting at point $x = x_0$, so that we use the delta function $w(x) = \delta(x - x_0)$, gives

$$u(x) = \int_{-\infty}^{\infty} \delta(\xi - x_0) G(\xi, x)\, d\xi = G(x_0, x).\square$$

**Example 5.11.5.** Consider the *integral equation*

$$\int_{-\infty}^{\infty} K(x-t)u(t)\, dt + \lambda u(x) = f(x)$$

where $K, f \in L^1(\mathbb{R})$ and the unknown function is $u(x)$. This is a *Fredholm integral equation with convolution kernel.* By the definition of convolution we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x-t)u(t)\, dt = (K * u)(x)$$

and by the Convolution Theorem (Theorem 4.11.7) we have

$$\mathscr{F}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x-t)u(t)\, dt\right\} = \mathscr{F}\{K * u\} = \mathscr{F}\{K\}\mathscr{F}\{u\} = \hat{K}\hat{u}.$$

So applying the Fourier transform to the integral equation gives

$$\sqrt{2}\pi \hat{K}(k)\hat{u}(k) + \lambda \hat{u}(k) = \hat{f}(k) \text{ or } \hat{u}(k) = \frac{\hat{f}(k)}{\sqrt{2\pi}\hat{K}(k) + \lambda}.$$

Therefore

$$u(k) = \mathscr{F}^{-1}\{u^{-1}(k)\} = \mathscr{F}^{-1}\left\{\frac{\hat{f}(k)}{\sqrt{2\pi}\hat{K}(k) + \lambda}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)e^{ikx}}{\sqrt{2\pi}\hat{K}(k) + \lambda}\, dk.\square$$

**Example 5.11.6. The Hilbert Transform.**

Consider the integral equation

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, dt = f_J(x)$$

where $f_H \in L^1(\mathbb{R})$ and the above integral is the "Cauchy principle value":

$$\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\varepsilon} \frac{f(t)}{x-t}\,dt + \int_{\varepsilon}^{\infty} \frac{f(t)}{x-t}\,dt\right).$$

The function $f_H$ is called the *Hilbert transform* of $f$ and is denoted $f_H(x) = \mathscr{H}\{f(t)\}$. First we let $g(x) = \sqrt{2/\pi}/x$ so that (based on Debnath and Mikusinski's claim at the top of page 275)

$$\hat{g}(k_= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} g(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x}\,dx = \frac{1}{\pi} i\pi\,\mathrm{sgn}(-k) = -i\,\mathrm{sgn}(k)$$

and the given integral equation becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt = \frac{1}{\pi}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} f(t)g(x-t)\,dt = f*g = f_H(x).$$

Applying the Fourier transform and the Convolution Theorem (Theorem 4.11.7) we have

$$\mathscr{F}\{f*g\} = \mathscr{F}\{f\}\mathscr{F}\{g\} = \hat{f}(k)\hat{g}(k) = \mathscr{F}\{f_H\} = \hat{f}_H(k),$$

or

$$\hat{f}(k) = \frac{\hat{f}_H(k)}{\hat{g}(k)} = \frac{\hat{f}_H(k)}{-i\,\mathrm{sgn}(k)} = i\,\mathrm{sgn}(k)\hat{f}_H(k).$$

Applying the inverse Fourier transform yields

$$\begin{aligned}
f(x) &= \mathscr{F}^{-1}\{i\,\mathrm{sgn}(k)\hat{f}_H(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\,\mathrm{sgn}(k)\hat{f}_H(k)e^{ikx}\,dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-\hat{g}(k))\hat{f}_H(k)e^{ikx}\,dx = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k)\hat{f}_H(k)e^{ikx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}\hat{g}(k)\hat{f}_H(k)e^{-ik\xi}\,d\xi \text{ where } \xi = -x \\
&= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k)\hat{f}_H(k)e^{-ik\xi}\,d\xi = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\hat{g}*\hat{f}_H)(k)e^{-ik\xi}\,d\xi \\
&= -\mathscr{F}^{-1}\{\mathscr{F}\{g\}*\mathscr{F}\{f_H\}\} = -\mathscr{F}^{-1}\{\mathscr{F}\{g*f_H\}\} \text{ by the Convolution Theorem} \\
&= -g*f_H = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-\xi)f_H(\xi)\,d\xi = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{2/\pi}}{x-\xi}f_(\xi)\,d\xi
\end{aligned}$$

$$= \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{f_H(\xi)}{x - \xi} \, d\xi = -\mathscr{H}\{f_H(\xi)\}.$$

That is, $f(x) = -\mathscr{H}\{f_H(\xi)\} = -\mathscr{H}\{\mathscr{H}\{f(t)\}\}$ and so (since $f$ is an arbitrary element of $L^2(\mathbb{R})$) the Hilbert transform $\mathscr{H}$ satisfies $\mathscr{H}^{-1} = -\mathscr{H}$ on $L^2(\mathbb{R})$. $\square$

*Revised: 2/10/2020*