Section 7.2. Basic Concepts and Equations of Classical Mechanics

Note. We review some classical mechanics, including Lagrange's equations of motion.

Note. Let $\vec{r}_i$ be the position of particle i and let $\vec{F}_i$ be the force on this particle. Then from Newton's Second Law of Motion,

$$m_i = \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i.$$

If there is function $V(\vec{r}, t)$ such that

$$F_i = -\nabla_i V$$

where $\nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right)$ and $\vec{r}_i = (x_i, y_i, z_i)$ then V is a *potential energy function.* If a force field is irrotational (i.e., $\nabla \times \vec{F} = \vec{0}$) and independent of time, it is a *conservative force field.*

Note. The kinetic energy of the ith particle is

$$T_i = \frac{1}{2}m_i|\vec{v}_i|^2 = \frac{1}{2}m_i\left(\frac{d\vec{r}_i}{dt}\right)\cdot\left(\frac{d\vec{r}_i}{dt}\right)$$

With momentum $\vec{p}_i = m_i\vec{v}_i$, Newton's Second Law is

$$-\nabla_i V = \vec{F}_i = \frac{d\vec{p}_i}{dt}.$$

Now

$$\frac{\partial T}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\sum_i T_i\right] = m_i\dot{x}_i = p_i^x.$$

So from Newton's Second Law,

$$\frac{d\vec{p}_i}{dt} = m_i\ddot{\vec{r}}_i = \left[\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right], \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{y}_i}\right], \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right]\right]$$

and so

$$0 = \left[\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right], \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{y}_i}\right], \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right]\right] + \underbrace{\left[\frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial y_i}, \frac{\partial V}{\partial x_i}\right]}_{\nabla_i V}. \qquad (*)$$

Note. The total energy E of a particle is the sum of its kinetic and potential energies, $E = T + V(\vec{r}, t)$. We <u>assume</u> V is independent of time. Then

$$\frac{dT}{dt} = \frac{d}{dt}\left[\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right] = \sum_i \frac{1}{2} m_i \left(\dot{\vec{v}}_i \cdot \vec{r}_i + \vec{r}_i \cdot \vec{v}_i\right) = \sum_i m_i(\vec{v}_i \cdot \dot{\vec{v}}_i)$$

and

$$\frac{dV(\vec{r}, t)}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} + \frac{\partial V}{\partial t} = \vec{v} \cdot \nabla V + 0 = \vec{v} \cdot \nabla V.$$

Therefore

$$\frac{dE}{dt} = \frac{DT}{dt} + \frac{dV}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{v} \cdot \nabla V = \vec{v} \cdot \left(\frac{d\vec{p}}{dt} + \nabla V\right) = 0.$$

So E is constant.

Definition. The *Lagrangian function* L of the above system of particles is the difference of the kinetic and potential energies:

$$L = T - V = L(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{(z)}_i).$$

Note/Definition. Since V is independent of time, $\dfrac{\partial L}{\partial \dot{x}_i} = \dfrac{\partial T}{\partial \dot{x}_i}$ and since T depends only on velocity and not on position $\dfrac{\partial L}{\partial x_i} = -\dfrac{\partial V}{\partial x_i}$. Therefore $(*)$ becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_i}\right] - \frac{\partial L}{\partial x_i} = 0,$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_i}\right] - \frac{\partial L}{\partial y_i} = 0,$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_i}\right] - \frac{\partial L}{\partial z_i} = 0.$$

These are the *Lagrange equations of motion.*

Revised: 4/19/2019