# Chapter I. Basic Ideas of Hilbert Space Theory

**Note.** The goal of this chapter is to introduce the idea of an infinite dimensional Hilbert space. In Section I.1 we review properties of vector spaces, in Section I.2 we review inner product spaces, and in Section I.3 we consider metric spaces. Hilbert spaces are introduced in Section I.4 and every separable Hilbert space is shown to be isomorphic to $\ell^2$. In Section I.5 awe introduce a state function $\Psi(t)$ which is an element of a Hilbert space and a wave function $\psi(x,t)$ as the description of a particle moving in one dimension in the presence of a potential well.

## Section I.1. Vector Spaces

**Note.** In this section we review several topics from sophomore level Linear Algebra (MATH 2010). Our *real* reason for this is to introduce the notation we use in this text for vectors and vector spaces.

**Definition I.1.1.** Any set $\mathcal{V}$ (the elements of which are called *vectors*) on which the operations of vector addition and multiplication by a scalar (the scalars are from some field $\mathsf{F}$) are defined and satisfy the axioms listed below is a *vector space* or *linear space* (or, less frequently, *linear manifold*). The operation of *vector addition* is a mapping form $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ where we denote the image of $(f,g) \in \mathcal{V} \times \mathcal{V}$ under this mapping is denoted $f + g \in \mathcal{V}$. The operation of *multiplication by scalar* $a \in \mathsf{F}$

is a mapping of $\mathsf{F} \times \mathcal{V}$ where the image of $(a, f) \in \mathsf{F} \times \mathcal{V}$ under the mapping is denoted $af \in \mathcal{V}$. These two operations of vector addition and scalar multiplication are required to satisfy the following, which we list as axioms. For all $f, g, h \in \mathcal{V}$ and for all $a, b \in \mathsf{F}$:

**Axiom 1.** $f + g = g + f$ (Commutivity of vector addition),

**Axiom 2.** $(f + g) + h = f + (g + h)$ (Commutivity of vector addition),

**Axiom 3.** There is a vector $\mathbf{0}$, called the *zero vector*, such that $g$ satisfies $f + g = f$ is and only if $g = \mathbf{0}$,

**Axiom 4.** $a(f + g) = af + ag$ (Distribution of scalar addition over scalar addition),

**Axiom 5.** $(a + b)f = af + bf$ (Distribution of scalar addition over scalar multiplication),

**Axiom 6.** $(ab)f = a(bf)$ (Distribution of scalar addition over scalar multiplication),

**Axiom 7.** $af = f$ where $a$ denotes the unit element (that is, the multiplicative identity) in the field.

We denote the vector space, along with these operations, as $\mathcal{V}$. We call $\mathcal{V}$ a *vector space over the field* $\mathsf{F}$. If $\mathsf{F} = \mathbb{R}$ or $\mathsf{F} = \mathbb{C}$ then the space is a *real vector space* or a *complex vector space*, respectively.

**Note.** Notice that we distinguish scalar 0 from vector $\mathbf{0}$ by using a bold faced font for the zero vector.

**Example.** $\mathbb{R}^n$ and $\mathbb{C}^n$ are examples of vector spaces where the vectors are $n$-tuples of scalars (and fixed $n \in \mathbb{N}$). Similarly, $\mathsf{F}^n$ is a vector space. Other examples of vector spaces are given in Exercises I.1.3 ($\mathcal{C}(\mathbb{R}^1)$, the infinite-dimensional vector space of all complex-valued continuous functions defined on $\mathbb{R}$) and I.1.7 ($\mathcal{P}_\infty$ the vector space of all polynomials, and $\mathcal{P}_n$ the vector space of all polynomials of degree at most $n$).

**Theorem I.1.1.** Every vector space $\mathcal{V}$ has only one zero vector $\mathbf{0}$, and each element $f$ of a vector space has one and only one additive inverse $(-f)$. For any $f \in \mathcal{V}$, we have $0f = \mathbf{0}$ and $(-1)f = (-f)$.

**Definition I.1.2.** The vectors $f_1, f_2, \ldots, f_n \in \mathcal{V}$ are *linearly independent* if for scalars $c_1, c_2, \ldots, c_n \in \mathsf{F}$, the equation $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = \mathbf{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$. A subset $S \subset \mathcal{V}$ is a *set of linearly independent vectors* if any finite number of different vectors from $S$ are linearly independent. The *dimension* of vector space $\mathcal{V}$ is the least upper bound (which can be finite or positive infinity) of the set of all integers $v$ for which there are $v$ linearly independent vectors in $\mathcal{V}$.

**Note.** The previous definition of "dimension" of $\mathcal{V}$ is a bit unconventional. It is more common to show that all bases of a vector space are of the same cardinality and to define this common cardinality as the dimension. We resolve Prugovečki's approach with the conventional approach for finite dimensional vector spaces in the next two theorems.

**Theorem I.1.2.** If the vector space $\mathcal{V}$ is $n$ dimensional, where $n \in \mathbb{N}$, then there is at least one set $f_1, f_2, \ldots, f_n$ of linearly independent vectors, and each vector $f \in \mathcal{V}$ can be expanded as $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$, there the coefficients $a_1, a_2, \ldots, a_n$ are uniquely determined by $f$.

**Definition I.1.3.** The (finite or infinite) set $S$ *spans* vector space $\mathcal{V}$ if every $f \in \mathcal{V}$ can be written as a linear combination $f = a_1 h_1 + a_2 h_2 + \cdots + a_n h_n$ where $h_1, h_2, \ldots, h_n \in S$ and $a_1, a_2, \ldots, a_n \in \mathsf{F}$. If $S$ is in addition a set of linearly independent vectors, then $S$ is a *basis* of $\mathcal{V}$.

**Note.** We now show that any two bases for a finite dimensional vector space are of the same size. To give a completely rigorous proof, we need a result not in Prugovečki. The following is from *Real Analysis with an Introduction to Wavelets*, D. Hong, J. Wang, and R. Gardner, Elsevier Press (2005). It appears as Lemma 5.1.1 in the book and holds in an arbitrary field.

**Lemma I.1.A.** Consider the homogeneous system of equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
\vdots & & & & \ddots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}$$

with coefficients $a_{ij}$ ($1 \le i \le m$, $1 \le j \le n$) and unknowns $x_k$ ($1 \le k \le n$) from field $\mathbb{F}$. If $n > m$ then the system has a nontrivial solution (that is, a solution $x_1, x_2, \ldots, x_n$ where $x_k \ne 0$ for some $1 \le k \le n$).

**Theorem I.1.3.** If the set $\{g_1, g_2, \ldots, g_n\}$ is a basis of $n$-dimensional vector space $\mathcal{V}$ (where $n \in \mathbb{N}$), then $m = n$. That is, all bases of an $n$-dimensional vector space are of the same size $n$.

**Definition I.1.4.** A subset $\mathcal{V}_1$ of a vector space $\mathcal{V}$ is a *vector subspace* (or *linear subspace*) of $\mathcal{V}$ if it is closed under the vector operations; that is, if $f + g \in \mathcal{V}_1$ and $af \in \mathcal{V}_1$ whenever $f, g \in \mathcal{V}_1$ and for any scalar $a$. A vector subspace $\mathcal{V}_1$ of $\mathcal{V}$ is *nontrivial* if it is different from $\mathcal{V}$ and different from $\{\mathbf{0}\}$.

**Note.** We now show that if $\mathcal{V}$ is an $n$-dimensional vector space (where $n \in \mathbb{N}$) with real scalars, then $\mathcal{V}$ is isomorphic to $\mathbb{R}^n$. If $\mathcal{V}$ is $n$-dimensional with complex scalars, then $\mathcal{V}$ is isomorphic to $\mathbb{C}^n$ More generally (though not shown in Prugovečki) is that $n$-dimensional vector space $\mathcal{V}$ with scalars from field $\mathsf{F}$ is isomorphic to $\mathsf{F}^n$. I call this result the "Fundamental Theorem of Finite Dimensional Vector Spaces" (a label introduced in the book *Real Analysis with an Introduction to Wavelets*, mentioned above). In fact, a similar result holds for infinite dimensional vector spaces (with some additional hypotheses), as we will see in Theorem I.4.7.

**Definition I.1.5.** Two vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ over the same field are *isomorphic* if there is a one to one mapping of $\mathcal{V}_1$ onto $\mathcal{V}_2$ which has the properties that if $f_2$ and $g_2$ (where $f_2, g_2 \in \mathcal{V}_2$) are the images of $f_1$ and $g_1$ (where $f_1, g_1 \in \mathcal{V}_1$), respectively, then for any scalar $a$, $af_2$ is the image of $af_1$ and $f_2 + g_2$ is the image of $f_1 + g_1$.

**Note.** A simpler classification of the isomorphism of Definition I.1.4 is that the mapping is a one to one and onto linear mapping. Now for our biggest result concerning finite dimensional vector spaces.

**Theorem I.1.4. The Fundamental Theorem of Finite Dimensional Vector Spaces.**
All complex (real) $n$-dimensional ($n \in \mathbb{N}$) vector spaces are isomorphic to the vector space $\mathbb{C}^n$ (or $\mathbb{R}^n$ in the case of real vector spaces).

**Note.** Since being isomorphic is an equivalence relation (see Exercise I.1.6), we see that any two $n$-dimensional vector spaces over field $\mathsf{F}$ are isomorphic to each other. Notice that it is meaningless to say that $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$ *as vector spaces*, since a vector space isomorphism is only defined between vector spaces over the same field. Notice Exercise I.1.2 and the use of the word "becomes."

**Note.** Other proofs of The Fundamental Theorem of Finite Dimensional Vector Spaces are available: (1) in my Linear Algebra (MATH 2010) notes which uses ordered bases (see `http://faculty.etsu.edu/gardnerr/2010/c3s3.pdf` and Theorem 3.3.A), and (2) in Theorem 5.4.9 of *Real Analysis with an Introduction to Wavelets* (see `http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-4.pdf`).

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