Section I.2. Euclidean (Pre-Hilbert) Spaces

Note. In this section we introduce vector spaces which admit an inner product. Such a vector space is a *Euclidean space* or *inner product space* (or less commonly, a *pre-Hilbert space*). We could choose to deal with real vector spaces and real valued inner products (as you do in Linear Algebra when you address dot products on $\mathbb{R}^n$), but instead we will deal with complex vector spaces with complex valued inner products. This is because of our intended applications to quantum mechanics. We adopt Prugovečki's notation that for $z = a + ib \in \mathbb{C}$ (where $a, b \in \mathbb{R}$), the complex conjugate is $z^* = a - ib$.

Definition I.2.1. An *inner product* (or *scalar product*) $\langle \cdot \mid \cdot \rangle$ on a complex vector space $\mathcal{V}$ is a mapping of the set $\mathcal{V} \times \mathcal{V}$ into the scalar field $\mathbb{C}$, where we denote the image of $(f, g) \in \mathcal{V} \times \mathcal{V}$ as $\langle f \mid g \rangle \in \mathbb{C}$, which satisfies the following:

1. $\langle f \mid f \rangle > 0$ for all $f \neq \mathbf{0}$,

2. $\langle f \mid g \rangle = \langle g \mid r \rangle^*$,

3. $\langle f \mid ag \rangle = a\langle f \mid g \rangle$, and

4. $\langle f \mid g + h \rangle = \langle f \mid g \rangle + \langle f \mid h \rangle$,

for all $f, g \in \mathcal{V}$ and for all $a \in \mathbb{C}$.

Note. Progovečki comments of page 18 that mathematicians prefer the notation (f, g) for an inner product ($\langle f, g\rangle$ is also a common notation) and that property (3) of Definition I.2.1 is often replace with $\langle af \mid g\rangle = a\langle f \mid g\rangle$. The notation we use, $\langle f \mid g\rangle$, was introduced by physicist P.A.M. Dirac in *The Principles of Quantum Mechanics*, Oxford University Press (1930) (a copy is available online at `http://digbib.ubka.uni-karlsruhe.de/volltexte/wasbleibt/57355817/57355817.pdf`, accessed 11/22/2018).

Theorem I.2.1. In a Euclidean space $\mathcal{E}$, the inner product $\langle f \mid g\rangle$ satisfies the relations:

(a) $\langle af \mid g\rangle = a^*\langle f \mid g\rangle$, and

(b) $\langle f + g \mid h\rangle = \langle f \mid h\rangle + \langle g \mid h\rangle$

for all $f, g, h \in \mathcal{E}$ and for every scalar a.

Example. Vector space $\mathbb{C}^n$ with inner product defined for $\alpha = [a_1, a_2, \ldots, a_n]^T, \beta = [b_1, b_2, \ldots, b_n]^T \in \mathbb{C}^n$ as

$$\langle \alpha \mid \beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n,$$

is a Euclidean space. We denote this space as $\ell^2(n)$.

Example. The previous example is a finite dimensional Euclidean space. An infinite dimensional Euclidean space is given by the vector space $\mathcal{C}^0_{(2)}(\mathbb{R})$ of all continuous complex-valued functions $f(x)$ defined on $\mathbb{R}$ which satisfy

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \text{ and } \lim_{x \to \pm\infty} f(x) = 0$$

where the inner product is

$$\langle f \mid g \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, dx.$$

This is, in fact, an inner product (see Exercise I.2.1); notice that it is not clear that $\langle f \mid g \rangle$ is finite (though it is) and it is not clear that $f + g$ satisfies the integrability condition (though it does).

Note. As you know, we will use inner products to induce norms. A first step in that direction is the following.

Theorem I.2.2. Schwarz-Cauchy Inequality.

Any two elements f, g of a Euclidean space $\mathcal{E}$ satisfies

$$|\langle f \mid g \rangle|^2 \leq \langle f \mid f \rangle \langle g \mid g \rangle.$$

Definition I.2.2. A mapping of $\mathcal{V}$ to $\mathbb{R}$, where we denote the image of f as $\|f\|$, is a *norm* if it satisfies the following:

1. $\|f\| > 0$ for $f \neq \mathbf{0}$,

2. $\|\mathbf{0}\| = 0$,

3. $\|af\| = |a|\|f\|$, and

4. $\|f + g\| \leq \|f\| + \|g\|$ (the Triangle Inequality),

for all $f, g \in \mathcal{V}$ and for all scalars $a \in \mathbb{C}$. A vector space which admits a norm is a *normed vector space* (or *normed linear space*).

Note. The next result shows that every Euclidean space $\mathcal{E}$ is in fact a normed vector space.

Theorem I.2.3. In a Euclidean space $\mathcal{E}$ with inner product $\langle f \mid g \rangle$, the real-valued function $\|f\| = \sqrt{\langle f \mid f \rangle}$ is a norm.

Note. In addition to using inner products to induce a norm, we can also (as happens with dot products in $\mathbb{R}^n$, as seen in Linear Algebra) use inner products to define orthogonality and projections.

Definition I.2.3. In a Euclidean space $\mathcal{E}$ two vectors f and g are *orthogonal*, denoted $f \perp g$, if $\langle f \mid g \rangle = 0$. Two subsets R and S of $\mathcal{E}$ are *orthogonal sets*, denoted $S \perp R$, if each vector in R is orthogonal to each vector in S. A set of vectors from $\mathcal{E}$ in which any two vectors are orthogonal is an *orthogonal system* of vectors. A vector vector f is *normalized* (or is a *unit vector*) if $\|f\| = 1$. An orthogonal system of vectors is an *orthonormal system* if each vector in the system is normalized.

Note. The following theorem makes use of the Gram-Schmidt Process. For notes on this process at the Linear Algebra level, see `http://faculty.etsu.edu/gardnerr/2010/c6s2.pdf`. For notes at the graduate level, see these notes from Theory of Matrices (MATH 5090): `http://faculty.etsu.edu/gardnerr/5090/notes/Chapter-2-2.pdf`.

Theorem I.2.4. If S is a finite of countably infinite set of vectors in a Euclidean space $\mathcal{E}$ and $\mathcal{V}$ is the vector subspace of $\mathcal{E}$ spanned by S, then there is an orthonormal system T of vectors which spans $\mathcal{V}$; that is, for which $\text{span}(T) = \mathcal{V}$ (that is, the set of all linear combinations of elements of T; Prugovečki denotes the space of T as (T)). T is a finite set when S is a finite set.

Note. We now define an isomorphism between inner product spaces and prove a not-too-surprising result for finite dimensional inner product spaces.

Definition I.2.4. Two Euclidean spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ with inner products $\langle \cdot \mid \cdot \rangle_1$ and $\langle \cdot \mid \cdot \langle_2$, respectively, are *isomorphic* (or *unitarily equivalent*) if there is a mapping of $\mathcal{E}_1$ onto $\mathcal{E}_2$ such that if $f_1, g_1 \in \mathcal{E}_1$, $f_2 \in \mathcal{E}_2$ is the image of f_1, and $g_2 \in \mathcal{E}_2$ is the image of g_1 under the mapping then $f_1 + g_1$ is mapped to $f_2 + g_2$, af_1 is mapped to af_2 for all $a \in \mathbb{C}$, and $\langle f_1 \mid g_1 \rangle_1 = \langle f_2 \mid g_2 \rangle_2$. The mapping (which is an inner product isomorphism) is called a *unitary transformation*.

Theorem I.2.5. All complex Euclidean n-dimensional spaces are isomorphic to $\ell^2(n)$ and consequently mutually isomorphic.

Note. Of course, Theorem I.2.5 has an analogous result for real inner product spaces of dimension n.

Theorem I.2.6. A unitary transformation from Euclidean space $\mathcal{E}$, onto Euclidean space $\mathcal{E}_2$ has a unique inverse mapping which is a unitary transformation of $\mathcal{E}_2$ onto $\mathcal{E}_1$.

Note. A proof of Theorem I.2.6 is to be given in Exercise I.2.A.

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