Section I.3. Metric Spaces

Note. In this section we define a metric space and a complete metric space. The main result is the fact that every incomplete metric space can be embedded in a complete metric space (Theorem I.3.2).

Note. In order to do "analysis things," such as discuss continuity, sequences, series, and integrals, we need to take limits. This requires at least a topology, but in this book we will always have a metric (which induces the metric topology).

Definition I.3.1. If S is a set, a real valued function $d(\xi, \eta)$ on $S \times S$ is a *metric* if for any $\xi, \eta, \zeta \in S$:

1. $d(\xi, \eta) > 0$ if $\xi \neq \eta$,

2. $d(\xi, \xi) = 0$,

3. $d(\xi, \eta) = d(\eta, \xi)$, and

4. $d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$ (the Triangle Inequality).

A set S on which a metric is defined is a *metric space*.

Note. First, we address sequences in a metric space.

Definition I.3.2. An infinite sequence $\xi_1, \xi_2, \ldots$ in a metric space $\mathcal{M}$ is said to *converge* to $\xi \in \mathcal{M}$ if for any $\varepsilon > 0$ there is a $N(\varepsilon) > 0$ such that $d(\xi, \xi_n) < \varepsilon$ for all $n > N(\varepsilon)$. An infinite sequence $\xi_1, \xi_2, \ldots$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there is a $M(\varepsilon) > 0$ such that $d(\xi_m, \xi_n) < \varepsilon$ for all $m, n > M(\varepsilon)$.

Theorem I.3.1. If a sequence $\xi_1, \xi_2, \ldots,$ in a metric space $\mathcal{M}$ converges to some $\xi \in \mathcal{M}$ then its limit is unique, and the sequence is a Cauchy sequence.

Note. We now use Cauchy sequences to define completeness in a metric space.

Definition I.3.3. A metric space $\mathcal{M}$ is *complete* if every Cauchy sequence converges to an element of $\mathcal{M}$.

Example. The real numbers $\mathbb{R}$ are complete (by definition, $\mathbb{R}$ is a complete ordered field) with metric $d(x, y) = |x - y|$. The rational numbers $\mathbb{Q}$ under the same metric is not complete; a sequence of rational numbers which converges to $\sqrt{2}$ (as a sequence in $\mathbb{R}$) is Cauchy by Theorem I.3.1, but is not convergent in $\mathbb{Q}$ since $\sqrt{2} \not\in \mathbb{Q}$.

Definition I.3.4. A subset S of a metric space $\mathcal{M}$ is (*everywhere*) *dense* in $\mathcal{M}$ if for any given $\varepsilon > 0$ and any $\xi \in \mathcal{M}$, there is an element $\eta \in S$ for which $d(\xi, \eta) < \varepsilon$.

Note. $\mathbb{Q}$ is dense in $\mathbb{R}$. $\mathbb{Z}$ is not dense in $\mathbb{R}$ (nor in $\mathbb{Q}$). We now use the metric to define some topological ideas.

Definition I.3.5. If ξ is an element of a metric space $\mathcal{M}$, then the set of all points η satisfying the inequality $d(\xi, \eta) < \varepsilon$ for some $\varepsilon < 0$ is the ε *neighborhood* of ξ. If S is a subset of $\mathcal{M}$, a point $\zeta \in \mathcal{M}$ is an *accumulation point* (or *cluster point* or *limit point*) of S if every ε neighborhood of ζ contains a point of S. The set $\overline{S}$ consisting of all the cluster points of S is the *closure* of S. If $S = \overline{S}$ then S is a *closed set*.

Note. Subset S of $\mathcal{M}$ is (everywhere) dense in $\mathcal{M}$ if and only if $\overline{S} = \mathcal{M}$.

Definition I.3.6. A one to one mapping of metric space $\mathcal{M}$ into metric space $\tilde{\mathcal{M}}$ is *isometric* if it preserves distances; that is, $d_1(\xi, \eta) = d_2(\tilde{\xi}, \tilde{\eta})$ for any $\xi, \eta \in \mathcal{M}$ and $\tilde{\xi}, \tilde{\eta} \in \tilde{\mathcal{M}}$ where $\xi \mapsto \tilde{\xi}$ and $\eta \mapsto \tilde{\eta}$. A metric space $\mathcal{M}$ is *densely embedded* in metric space $\tilde{\mathcal{M}}$ if there is an isometric mapping of $\mathcal{M}$ into $\tilde{\mathcal{M}}$, and if the image $\mathcal{M}'$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is everywhere dense in $\tilde{\mathcal{M}}$.

Note. We now state a major result concerning embedding a given metric space into a complete metric space.

Theorem I.3.2. Every incomplete metric space $\mathcal{M}$ can be embedded in a complete metric space $\tilde{\mathcal{M}}$, called the *completion* of $\mathcal{M}$.

Note. The proof is similar to the construction of the real numbers based on the completion of the rationals. In this construction, a real number is defined as an equivalence class of Cauchy sequences of rationals. Prugovečki refers to this as "Cantor's construction." This same result is also seen in our Introduction to Topology (MATH 4357/5357). See Theorem 43.7 of my class notes at: `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-43.pdf`.

Note. Before proving Theorem I.3.2, we need to introduce some notation and definitions.

Definition. Let $\mathcal{M}$ be a metric space. Let $\tilde{\mathcal{M}}_S$ be the set of all Cauchy sequences in $\mathcal{M}$. The Cauchy sequences $\tilde{\xi}' = \{\xi_1', \xi_2', \ldots\}$ and $\tilde{\xi}'' = \{\xi_1'', \xi_2'', \ldots\}$ of elements of $\mathcal{M}$ are *equivalent* if $\lim_{n\to\infty} d(\xi_n', \xi_n'') = 0$. (By Exercise I.3.1, equivalence is an equivalence relation on $\tilde{\mathcal{M}}_S$.) The family of all equivalence classes on $\tilde{\mathcal{M}}_S$ with respect to this equivalence relation is denoted $\tilde{\mathcal{M}}$. We denote both the sequence $\tilde{\xi} \in \tilde{\mathcal{M}}_S$ and the equivalence class in $\tilde{\mathcal{M}}$ containing $\tilde{\xi}$ as "$\tilde{\xi}$."

Definition. Define a real valued function d_S on $\tilde{\mathcal{M}}_S \times \tilde{\mathcal{M}}_S$ where for $\tilde{\xi} = \{\xi_1, \xi_2, \ldots\}$ and $\tilde{\eta} = \{\eta_1, \eta_2, \ldots\}$ we have $d_S(\tilde{\xi}, \tilde{\eta}) = \lim_{n\to\infty} d(\xi_n, \eta_n)$.

Note. It is to be shown in Exercise I.3.2 that for $\tilde{\xi}, \tilde{\eta} \in \tilde{\mathcal{M}}_S$ as above, we have

$$|d(\xi_m, \eta_m) - d(\xi_n, \eta_n)| \leq d(\xi_m, \xi_n) + d(\eta_m, \eta_n).$$

Since $\tilde{\xi}$ and $e\tilde{t}a$ are Cauchy sequences, then the sequence $\{d(\xi_1, \eta_1), d(\xi_2, \eta_2), \ldots\}$ of real numbers is Cauchy and hence convergent. So the limit in the previous definition is defined and hence d_S is defined.

Definition. Define a real valued function, denoted d_E, on $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$ where for equivalence classes $\tilde{\xi}, \tilde{\eta} \in \tilde{\mathcal{M}}$ we have $d_E(\tilde{\xi}, \tilde{\eta}) = d_S(\tilde{\xi}, \tilde{\eta})$ where on the right hand side of this equation $\tilde{\eta} \in \mathcal{M}_S$ and $\tilde{\eta} \in \mathcal{M}_S$ are <u>elements</u> (or "representatives") of equivalence classes $\tilde{\xi}$ and $\tilde{\eta}$, respectively, and on the left hand side $\tilde{\xi}, \tilde{\eta} \in \tilde{\mathcal{M}}$ are equivalence classes. (Prugovečki denotes both d_E and d_S as "d_S," making reading parts of this section a challenge.)

Note. We need to show that d_E in the precious definition is well-defined (that is, is independent of the choice of the representatives used). Let $\tilde{\xi}' \sim \tilde{\xi}''$ and $\tilde{\eta}' \sim \tilde{\eta}''$ be equivalent elements of $\mathcal{M}_S$ under the equivalence relation $\sim$ on $\tilde{\mathcal{M}}_S$ (so here they are sequences). By Exercise I.3.3,

$$|d(\xi_n', \eta_n') - d(\xi_n'', \eta_n')| \leq d(\xi_n', \xi_n'').$$

Since $\tilde{\xi}_n' \sim \tilde{\xi}_n''$ then by definition of "$\sim$" (the equivalence relation on $\tilde{\mathcal{M}}_S$) the left hand side of this inequality approaches 0 as $n \to \infty$ and so $d_S(\tilde{\xi}', \tilde{\eta}') = d_S(\tilde{\xi}'', \tilde{\eta}')$. Similarly we can show $d_S(\tilde{\xi}'', \tilde{\eta}') = d_S(\tilde{\xi}'', \tilde{\eta}'')$. Therefore $d_S(\tilde{\xi}', \tilde{\eta}') = d_S(\tilde{\xi}'', \tilde{\eta}'')$ and so $d_E(\tilde{\xi}, \tilde{\eta})$ (where $\tilde{\xi}$ and $\tilde{\eta}$ are equivalence classes in $\tilde{\mathcal{M}}_S$) is well-defined and independent of the representatives of equivalence classes $\tilde{\xi}$ and $\tilde{\eta}$ in the definition of d_E on $\tilde{\mathcal{M}}$.

Note. The function d_E on $\tilde{\mathcal{M}}$ is in fact a metric, as is to be shown in Exercise I.3.4. So $\tilde{\mathcal{M}}$ equipped with d_E is a metric space and the elements of $\tilde{\mathcal{M}}$ are equivalence classes of Cauchy sequences of elements of $\mathcal{M}$. We next show that $\tilde{\mathcal{M}}$ is a complete metric space; it is the completion of $\mathcal{M}$. That is, we prove Theorem I.3.2.

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