Section I.4. Hilbert Spaces

Note. A Hilbert space is a complete inner product space. The inner product induces a norm and this is how we can address completeness. The "interesting" Hilbert spaces are infinite dimensional Euclidean spaces. We saw in Theorem I.2.5 that every n-dimensional complex Euclidean space is isomorphic to $\ell^2(n)$. We'll see a similar result in Theorem I.4.7 in which we see that every complex infinite-dimensional "separable" Hilbert space is isomorphic to $\ell^2(\infty)$ (this is the Fundamental Theorem of Infinite Dimensional Vector Spaces).

Definition. A complete normed space is a *Banach space*. A Euclidean space (or an "inner product space") is a *Hilbert space* if it is complete with respect to the norm induced by the inner product.

Note. Technically, we need a metric to discuss completeness, not a norm. But in an inner product space with inner product $\langle \cdot \mid \cdot \rangle$ we have the norm $\|f\| = \sqrt{\langle f \mid f \rangle}$ (by Theorem I.2.3) and the metric $d(f,g) = \|f - g\|$ (by Exercise I.4.1).

Example. We saw in the second example of Section I.2 that the vector space $\mathcal{C}^0_{(2)}(\mathbb{R})$ of all continuous complex-valued functions $f(x)$ defined on $\mathbb{R}$ which satisfy

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \text{ and } \lim_{x \to \pm\infty} f(x) = 0$$

is an inner product space where $\langle f \mid g \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, dx$. However, there is a Cauchy sequence in $\mathcal{C}^0_{(2)}(\mathbb{R})$ that does not converge to an element of $\mathcal{C}^0_{(2)}(\mathbb{R})$.

Consider

$$f_n(x) = \begin{cases} 1 & \text{for } |x| \le a \\ \exp(-n^2(|x|-a)^2) & \text{for } |x| > a \end{cases}.$$

By Exercise I.4.2, the sequence $\{f_n(x)\}$ is Cauchy. By Exercise I.4.A, $f_n \to f$ where $f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$ and the convergence is with respect to the metric induced by the inner product on $\mathcal{C}^0_{(2)}(\mathbb{R})$.

Definition I.4.1. Euclidean space $\mathcal{E}$ can be *densely embedded* in Hilbert space $\mathcal{H}$ if there is a one to one mapping of $\mathcal{E}$ into $\mathcal{H}$ such that the image $\mathcal{E}'$ of $\mathcal{E}$ is everywhere dense in $\mathcal{H}$, and the mapping represents an isomorphism between the Euclidean spaces $\mathcal{E}$ and $\mathcal{E}'$.

Note. We saw in Theorem I.3.2 that every incomplete metric space can be densely embedded in a complete metric space. We now show a similar result for Euclidean spaces.

Theorem I.4.1. Any incomplete Euclidean space $\mathcal{E}$ can be densely embedded in a Hilbert space.

Note. A result similar to Theorems I.3.2 and I.4.1 holds for normed spaces. That is, every incomplete normed linear space can be densely embedded in a complete normed linear space (i.e., in a Banach space). See Exercise I.4.5.

Note. Prugovečki state (page 32): "In quantum mechanics we deal at present almost exclusively with a special class of Hilbert spaces which are called separable." But the real reason to consider separable Hilbert spaces is that they are the type of Hilbert spaces which have an orthonormal basis (as we'll see in Theorem I.4.5).

Definition I.4.2. The Euclidean space $\mathcal{E}$ is *separable* if there is a countable everywhere dense subset of vectors of $\mathcal{E}$.

Note. Unless stated otherwise, Prugovečki means complex Hilbert space when he uses the term "Hilbert space."

Theorem I.4.2. Every subspace of a separable Euclidean space is a separable Euclidean space.

Note. We'll see in Theorem I.4.7 that any two separable Hilbert spaces are isomorphic (we will call this the Fundamental Theorem of Infinite Dimensional Vector Spaces). The following theorem gives us an example of such a space.

Theorem I.4.3. The set $\ell^2(\infty)$ of all one-column complex matrices α with countable number of elements, $\alpha = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$ for which $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ becomes a separable Hilbert space, also denoted $\ell^2(\infty)$, if the vector operations are defined by

$$\alpha + \beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \end{bmatrix}, \text{ and } a\alpha = a \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} aa_1 \\ aa_2 \\ \vdots \end{bmatrix}$$

for any scalar $a \in \mathbb{C}$, and the inner product is defined by $\langle \alpha \mid \beta \rangle = \sum_{k=1}^{\infty} a_k^* b_k$.

Note. We now adopt Prugovečki's notation concerning spans of sets. If S is a set of vectors from an infinite-dimensional Euclidean space, then the vector space spanned by set S is denoted (S). The (topologically) closed vector space spanned by set S is denoted $[S]$. More formally, we have the following.

Definition I.4.3. The *vector space* (or *linear manifold*) (S) *spanned by the subset* S of a Euclidean space $\mathcal{E}$ is the smallest subspace of $\mathcal{E}$ containing S (that is, if $\mathcal{V}$ is a subspace of $\mathcal{E}$ containing set S then $(S) \subset \mathcal{V}$). The *closed vector space* $[S]$ *spanned by* S is the smallest (topologically) closed subspace of $\mathcal{E}$ containing set S.

Note. If $S \subset \mathcal{E}$ where $\mathcal{E}$ is a finite dimensional vector space, then $(S) = [S]$ (see Exercise I.4.8 in which it is to be shown that every finite dimensional Euclidean space is a separable Hilbert space). The next result gives the relationship between (S) and $[S]$ is an infinite dimensional Euclidean space.

Theorem I.4.4. The subspace (S) of the Euclidean space $\mathcal{E}$ spanned by set S is identical with the set of all finite linear combinations $a_1f - 1 + a_2f_2 + \cdots + a_nf_n$ of vectors from S. That is,

$$(S) = \{a_1f_1 + a_2f_2 + \cdots + a_nf_n \mid f_1, f_2, \ldots, f_n \in S, a_1, a_2, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N}\}.$$

The closed linear subspace $[S]$ spanned by S is equal to the topological closure $\overline{(S)} = (S)$.

Note. The proof of Theorem I.4.4 is to be given in Exercise I.4.9.

Definition I.4.4. An orthonormal system S of vectors in a Euclidean space $\mathcal{E}$ is an *orthonormal basis* (or a *complete orthonormal system*) in the Euclidean space $\mathcal{E}$ is the closed linear space $[S]$ spanned by S equals the entire Euclidean space; that is, if $[S] = \mathcal{E}$.

Note. In a finite dimensional (complex) Euclidean space (which we know to be isomorphic to $\ell^2(n)$, where n is the dimension, by Theorem I.2.5), an orthonormal basis is given by the standard basis $e_1, e_2, \ldots, e_n$. However, in infinite dimensions such as in $\ell^2(\infty)$, the set of vectors $\{e_n \mid n \in \mathbb{N}\}$ where the mth component of e_n is δ_{mn} is not a basis of $\ell^2(\infty)$. Notice that $\alpha = [1, 1/2, 1/3, \ldots]^T \in \ell^2(\infty)$ (since, as a sequence, it is square summable), but α cannot be written as a (finite) linear combination of the e_n's. In a vector space, a basis is a linearly independent spanning set (see Definition I.1.3); the term "span" requires finite linear combinations. However, we have $\alpha = \sum_{n=1}^{\infty} \frac{1}{n} e_n$, but this is not a linear combination but instead is a series (and hence a limit). Limits require metrics (or, equivalently, norms). In fact, $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis (in the sense of Definition I.4.4) but not a vector space basis (in the sense of Definition I.1.3).

Note. An alternative approach to the concept of an orthonormal basis, is the idea of a Schauder basis (in contrast to a Hamel basis) in a vector space with a metric. For more details, see my online notes on "Groups, Fields, and Vector Spaces" at `http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-1.pdf`. The next result shows that the condition of separability is equivalent to the existence of a countable orthonormal basis.

Theorem I.4.5. A Euclidean space $\mathcal{E}$ is separable if and only if there is a countable orthonormal basis in $\mathcal{E}$.

Note. The next result gives necessary and sufficient conditions for an orthonormal system to be a basis. The result is stated and proved for infinite dimensional spaces but also holds for finite dimensional spaces.

Theorem I.4.6. Each of the following is a necessary and sufficient condition for a countable orthonormal system $T = \{e_1, e_2, \ldots\}$ to be a basis in a separable Hilbert space $\mathcal{H}$.

(a) The only vector f satisfying the relations $\langle e_k \mid f\rangle = 0$ for all $k \in \mathbb{N}$ is the zero vector, $\mathbf{0}$.

(b) For any vector $f \in \mathcal{H}$, $\lim_{n\to\infty} \|f - \sum_{k=1}^{n} \langle e_k \mid f\rangle e_k\| = 0$ or $f = \sum_{k=1}^{\infty} \langle e_k \mid f\rangle e_k$. The $\langle e_k \mid f\rangle$ are *Fourier coefficients* of f with respect to basis T.

(c) Any two vectors $f, g \in \mathcal{H}$ satisfy Parseval's relation $\langle f \mid g\rangle = \sum_{l=1}^{\infty} \langle f \mid e_k\rangle\langle e_k \mid g\rangle$.

(d) For any $f \in \mathcal{H}$, $\|f\| = \sum_{k=1}^{\infty} |\langle e_k \mid f\rangle|^2$.

Note. We need a preliminary lemma before presenting the proof.

Lemma I.4.1. For any given vector f in a Euclidean space $\mathcal{E}$ (not necessarily separable) and any countable system $\{e_1, e_2, \ldots\}$ in $\mathcal{E}$, the sequence $\{f_1, f_2, \ldots\}$ of vectors, $f_n = \sum_{k=1}^{n} \langle e_k \mid f \rangle e_k$ is a Cauchy sequence, and the Fourier coefficients $\langle e_k \mid f \rangle$ satisfy Bessel's inequality $\|f_n\| = \sum_{k=1}^{n} |\langle e_k \mid f \rangle|^2 \leq \|f\|^2$.

Note. We are now equipped to prove Theorem I.4.6.

Note. We now have the background to prove that all infinite dimensional separable Hilbert spaces are isomorphic. We elevate this to the status of "Fundamental Theorem of Infinite Dimensional Vector Spaces." There are infinite dimensional vector spaces which are not separable (consider $\mathbb{R}^\infty$; though we need a metric on $\mathbb{R}^\infty$ to discuss density). so the next theorem does not classify *all* infinite dimensional vector spaces.

Theorem I.4.7. Fundamental Theorem of Infinite Dimensional Vector Spaces.
All complex infinite-dimensional separable Hilbert spaces are isomorphic to $\ell^2(\infty)$, and consequently are mutually isomorphic.

Note. Prugovečki says (page 42) that the Fundamental Theorem of Infinite Dimensional Vector Spaces "provides the basis of the equivalence of Heisenberg's matrix formulation and Schroedinger's wave formulation of quantum mechanics."

Theorem I.4.8. Let $\mathcal{E}$ be a separable Euclidean space with an orthonormal basis $\{e_1, e_2, \ldots\}$ and let $\mathcal{E}'$ be a Euclidean space. If there is a unitary transformation from $\mathcal{E}$ to $\mathcal{E}'$ (that is, $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic inner product spaces) and if e_n transforms to e'_n, then $\{e'_1, e'_2, \ldots\}$ is an orthonormal basis in $\mathcal{E}'$.

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