Chapter II. Measure Theory and Hilbert Spaces of Functions

Note. In this chapter we explore abstract measure, integration, and additional Hilbert space theory. In Sections II.5 and II.7 we consider applications to quantum physics.

Section II.1. Measurable Spaces

Note. In this section we consider Boolean algebras and Boolean σ algebras of sets and use these to define the Borel sets in $\mathbb{R}^n$. We also introduce the idea of a monotone class of sets and give conditions under which a monotone class and Boolean σ algebra are equal.

Definition II.1.1. A nonempty class $\mathscr{K}$ of subset of a set $\mathscr{X}$ is called a *Boolean algebra* (or *field* or *additive class*) if the following hold:

1. $R \cup S \in \mathscr{K}$ whenever $R, S \in \mathscr{K}$, and

2. $R' = \mathscr{X} \setminus R \in \mathscr{K}$ whenever $R \in \mathscr{K}$.

The class $\mathscr{K}$ of subset of $\mathscr{X}$ is a *Boolean σ algebra* (or σ *field*) if in addition to being a Boolean algebra it has the property that $\cup_{k=1}^{\infty} S_n \in \mathscr{K}$ whenever $S_1, S_2, \ldots \in \mathscr{K}$.

Note. The collection of sets which we measure will form a Boolean σ algebra. We now state some elementary properties.

Theorem II.1.1. If the class $\mathscr{K}$ of subsets of a set $\mathscr{X}$ is a Boolean algebra, then

(a) the entire set $\mathscr{X}$ and the empty set $\varnothing$ belong to $\mathscr{K}$,

(b) the intersection $R \cap S$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$, and

(c) the difference $R \setminus S$ and symmetric difference $R \triangle S = (R \setminus S) \cup (S \setminus R)$ belongs to $\mathscr{K}$ whenever $R, S \in \mathscr{K}$.

Note. Before proving Theorem II.1.1, we need some very elementary results from set theory. You should be familiar with the following two lemmas.

Lemma II.1.1. If $\mathscr{F}$ is a family of sets and R is any given set, then

$$R \cap (\cup_{S \in \mathscr{F}} S) = \cup_{S \in \mathscr{F}} (R \cap S).$$

Lemma II.1.2. DeMorgan's Laws.

If $\mathscr{F}$ is a family of subsets of a set $\mathscr{X}$, and if for any given set S we denote by $S' = \mathscr{X} \setminus S$ the *complement* of S with respect to $\mathscr{X}$, then

$$(\cup_{S \in \mathscr{F}} S)' = \cap_{S \in \mathscr{F}} S' \text{ and } (\cap_{S \in \mathscr{F}} S)' = \cup_{S \in \mathscr{F}} S'.$$

Note. We can now prove Theorem II.1.1.

Theorem II.1.2. For any given nonempty family $\mathscr{F}$ of subset of a set $\mathscr{X}$ there is a unique smallest Boolean algebra $\mathscr{A}(\mathscr{F})$ and a unique smallest Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{F})$ containing $\mathscr{F}$. That is, if $\mathscr{A}$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}(\mathscr{F}) \subset \mathscr{A}$ and if $\mathscr{A}_\sigma$ is a Boolean algebra containing $\mathscr{F}$ then $\mathscr{A}_\sigma(\mathscr{F}) \subset \mathscr{A}_\sigma$. $\mathscr{A}(\mathscr{F})$ and $\mathscr{A}_\sigma(\mathscr{F})$ are called, respectively, the *Boolean algebra* and the *Boolean σ algebra generated* by the family $\mathscr{F}$.

Note. We denote the set of all intervals in $\mathbb{R}^n$, including degenerate intervals consisting of only one point and $\varnothing$, as $\mathscr{I}^n$. So the nonempty elements of $\mathscr{I}^n$ are of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. We are interested in the Boolean algebra on $\mathbb{R}^n$ generated by $\mathscr{I}^n$.

Theorem II.1.3. The family $\mathscr{B}_0^n$ of all finite unions

$$I_1 \cup I_2 \cup \cdots \cup I_k \text{ where } I_1, I_2, \ldots, I_n \in \mathscr{I}^n \text{ and } k \in \mathbb{N}$$

of intervals in $\mathscr{I}^n$ is identical to the Boolean algebra $\mathscr{A}(\mathscr{I}^n)$.

Note. By Exercise II.1.5, we also have that $\mathscr{B}_0^n = \mathscr{A}(\mathscr{I}^n)$ equals the family of all *disjoint* unions of intervals.

Definition. A subset of $\mathbb{R}^n$ in the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{I}^n)$ generated by $\mathscr{I}^n$ is a *Borel set* in n-dimensions.

Note. We cannot extend Theorem II.1.3 to countable unions and Boolean σ algebras. For example there is a Borel set in $\mathbb{R}$, namely $\mathbb{R} \setminus \mathbb{Q}$, that is not a countable union of intervals.

Note. Recall from Real Analysis 1 that the cardinality of the Borel sets on $\mathbb{R}$ is $\aleph_1$ and the cardinality of the power set of $\mathbb{R}$ is $\aleph_2$ (assuming the Continuum Hypothesis), so that there are subsets of $\mathbb{R}$ that are no Borel sets (in fact, there are Lebesgue measurable sets which are not Borel sets). See my online notes on "The Cardinality of the Set of Lebesgue Measurable Sets" at `http://faculty.etsu.edu/gardnerr/5210/notes/Cardinality-of-M.pdf`. However, as in the setting of $\mathbb{R}$, the Borel sets in $\mathbb{R}^n$ include all open and closed subsets of $\mathbb{R}^n$.

Theorem II.1.4. Every open and every closed set in the Euclidean space $\mathbb{R}^n$ is a Borel set.

Note. We now consider monotone classes of sets. This topic is not covered in Real Analysis, but is required in probability theory. See my online notes on Measure Theory Based Probability of "Lebesgue-Stieltjes Measure and Distribution Functions" at `http://faculty.etsu.edu/gardnerr/Probability/notes/Prob-1-4.pdf` and "Independent Random Variables" at `http://faculty.etsu.edu/gardnerr/Probability/notes/Prob-4-8.pdf`.

Definition II.1.2. An infinite sequence $S_1, S_2, \ldots$ of sets is called *monotonically increasing* if $S_1 \subset S_2 \subset \cdots$, and is called *monotonically decreasing* if $X_1 \supset S_2 \supset \cdots$. For a monotonically increasing sequence of sets we write $\lim_{k\to\infty} S_k = \cup_{k=1}^{\infty} S_k$ and for monotonically decreasing sequence of sets we write $\lim_{k\to\infty} S_k = \cap_{k=1}^{\infty} S_k$. A nonempty class $\mathscr{K}$ of subsets of a set $\mathscr{X}$ is called a *monotone class* if every monotone sequence $S_1, S_2, \ldots \in \mathscr{K}$ has its limit in $\mathscr{K}$.

Note. Very similar to the proof of Theorem II.1.2 for a Boolean algebra or Boolean σ algebra generated by a family of sets, we can prove the following concerning monotone class.

Theorem II.1.5. If $\mathscr{F}$ is a nonempty family of subsets of a set $\mathscr{X}$, there is a unique smallest monotone class $\mathfrak{M}(\mathscr{F})$ containing $\mathscr{F}$, which is called the monotone class *generated* by $\mathscr{F}$.

Note. We now give two close relationships between a Boolean σ algebra and a monotone class.

Theorem II.1.6. Every Boolean σ algebra is a monotone class, and every Boolean algebra which is a monotone class is a Boolean algebra.

Theorem II.1.7. If $\mathscr{A}$ is a Boolean algebra and $\mathfrak{M}(\mathscr{A})$ is the monotone class generated by $\mathscr{A}$, then $\mathfrak{M}(\mathscr{A})$ is identical with the Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by the family $\mathscr{A}$ of sets.

Note. We now define a measurable space exactly as in Real Analysis 1; see my online notes for "Measures and Measurable Sets" at `http://faculty.etsu.edu/gardnerr/5210/notes/17-1.pdf`.

Note. In the next section we require more structure on a measurable space and define a measure space.

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