Section II.2. Measures and Measure Spaces

Note. In this section we define measure and measurable space. We define an outer measure and the measure induced by an outer measure (with reference to the Carathéodory splitting condition). We also consider Cartesian products of measure spaces.

Definition. A set function F defined on a family $\mathscr{K}$ of sets into $\mathbb{R}$ (or $\mathscr{C}$) is *additive* if for any disjoint $S_1, S_2 \in \mathscr{K}$ we have $F(S_1 \cup S_2) = F(S_1) + F(S_2)$. Set function F is σ *additive* or *countably additive* if for any countable disjoint sequence of sets $S_1, S_2, \ldots \in \mathscr{K}$ we have $F\left(\cup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} F(S_k)$.

Definition II.2.1. A *measure* μ is an extended real-valued set function whose domain of definition is a Boolean σ algebra $\mathscr{A}$ and which satisfies the following:

1. $\mu(\varnothing) = 0$,

2. $\mu(S) \geq 0$ for all $S \in \mathscr{A}$, and

3. $\mu\left(\cup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n)$ if the sets $S_1, S_2, \ldots$ are disjoint.

Note. Progovečki defines "formal operations" on the extended real numbers $\mathbb{R} \cup \{\infty\}$: $a + \infty = \infty$, $a \cdot \infty = \infty$ if $a > 0$, $0 \cdot \infty = 0$, and $a/\infty = 0$ for all $a \in \mathbb{R} \cup \{\infty\}$ (see page 67). These last two are questionable…

Definition II.2.2. A *measure space* $(\mathscr{X}, \mathscr{A}, \mu)$ is a measurable space $(\mathscr{X}, \mathscr{A})$ for which a measure μ is defined on the Boolean σ algebra $\mathscr{A}$. For any $S \in \mathscr{A}$, the number $\mu(S)$ is the *measure of set* S. If $\mu(S) < \infty$ then S has *finite measure*. If the set S is a countable union $\cup_{n=1}^{\infty} S_n$ of sets S_n of finite measure, then the measure $\mu(S)$ of S is σ *finite*. If $\mu(\mathscr{X}) < \infty$ then μ is a *finite measure*, and if the measure $\mu(\mathscr{X})$ is σ finite then μ is a σ *finite measure*.

Note. A probability measure μ_{p} on a measurable space $(\mathscr{X}, \mathscr{A})$ where $\mu_{\text{p}}(S)$ represents the probability that an event (an element of $\mathscr{X}$) will be in set S. So we must have $\mu_{\text{p}}(\mathscr{X}) = 1$. The most common probability measure we will use is P_t defined on $\mathscr{B}_0^1$ (the family of all finite unions of intervals in $\mathbb{R}$; see Theorem II.1.3) as $P_t(I) = \int_I |\psi(x,t)|^2\, dx$ for interval I and $P_t(B) = P_t(I_1) + P_t(I_2) + \cdots + P_t(I_k)$ for $B = \cup_{\ell=1}^{k} I_\ell \in \mathscr{B}_0^1$, where $\psi(x,t)$ is the state of a system and $P_t(B)$ is the probability of finding the system within B at time t.

Definition II.2.3. An extended real-valued set function F is *continuous from above* (respectively, *below*) if for every monotonically decreasing (respectively, increasing) sequence $S_1, S_2, \ldots$ of sets from the domain of definition of F we have $F(\lim_{k\to\infty} S_k) = \lim_{k\to\infty} F(S_k)$ whenever $F(\lim_{k\to\infty} S_k)$ is defined and, in the case that $S_1, S_2, \ldots$ is decreasing, whenever $|F(S_n)| < \infty$ for at least one value of n.

Note. In the case of $S_1, S_2, \ldots$ decreasing, if we do not have $|F(S_n)| < \infty$ for at least one value of n, then we may not have $F(\lim_{k\to\infty} S_k) = \lim_{k\to\infty} F(S_k)$, as is to be shown in Exercise II.2.4 (consider F on $\mathscr{B}_0^1$ where $F([a,b]) = b - a$ and $S_n = [n, \infty)$). With the finiteness requirement in Definition II.2.3, we can prove the following.

Theorem II.2.1. Every measure is continuous from above and below.

Theorem II.2.2. Every finite, nonnegative, additive set function F on a Boolean σ algebra $\mathcal{A}$ and satisfying $F(\varnothing) = 0$, which is either continuous from below at every $R \in \mathcal{A}$ or continuous from above at $\varnothing \in \mathcal{A}$, is necessarily also σ additive or "countably additive" (i.e., μ is a measure).

Note. Prugovečki observes that it would be "convenient" if we could extend a measure on a Boolean algebra $\mathscr{A}$ to a Boolean σ algebra $\mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$, getting the measure space $(\mathscr{X}, \mathscr{A}_\sigma(\mathscr{A}), \mu)$. This is addressed in the next theorem. We consider this in Real Analysis 2 (MATH 5220) where we consider measures induced by outer measures on a power set and set functions on a collection of sets (for a "nice" result, the set function needs to be a premeasure and the collection of sets needs to be a semiring). See my online notes for "The Carathéodory Measure Induced by an Outer Measure" (`http://faculty.etsu.edu/gardnerr/5210/notes/17-3.pdf`) and "The Carathéodory-Hahn Theorem: The Extension of a Premeasure to a Measure" (`http://faculty.etsu.edu/gardnerr/5210/notes/17-5.pdf`).

Theorem II.2.3. Let μ be a measure defined on the Boolean algebra $\mathscr{A}$ of subsets of a given set $\mathscr{X}$. The set function

$$\overline{\mu} = \inf \left\{ \sum_{k=1}^{\infty} \mu(S_k) \;\middle|\; R \subset \cup_{k=1}^{\infty} S_k, S_k \in \mathscr{A} \text{ for all } n \in \mathbb{N} \right\}$$

for $R \in \mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$, is a measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A}$: $\overline{\mu}(S) = \mu(S)$ for all $S \in \mathscr{A}$. If μ is a σ finite measure, then $\overline{\mu}$ is also σ finite, and $\overline{\mu}$ is the only measure on $\mathscr{A}_\sigma$ which coincides with μ on $\mathscr{A}$. The measure $\overline{\mu}$ is called the *extension* of μ.

Note. We now lay out a proof of Theorem II.2.3, but need some preliminary results and another definition.

Lemma II.2.A. Let $\mathscr{A}$ be a Boolean algebra on set $\mathscr{X}$ and let μ be a measure on $\mathscr{A}$. Define extended real-valued set function

$$\mu^+(R) = \inf \left\{ \sum_{k=1}^{\infty} \mu(S_k) \;\middle|\; R \subset \cup_{k=1}^{\infty} S_k, S_k \in \mathscr{A} \text{ for all } k \in \mathbb{N} \right\}$$

on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$. Sets $S_1, S_2, \ldots \in \mathscr{A}$ such that $R \subset \cup_{k=1}^{\infty} S_k$ are said to *cover* R. For any $R_1, R_2, \ldots \in \mathscr{G}_{\mathscr{X}}$ we have

$$\mu^+\left(\cup_{n=1}^{\infty} R_n\right) \leq \sum_{n=1}^{\infty} \mu^+(R_n).$$

Definition II.2.4. A nonnegative set function M defined on each subset $R \in \mathscr{G}_{\mathscr{X}}$ of set $\mathscr{X}$, for which $M(\varnothing) = 0$ and which is *countably subadditive*, i.e. for which

$$M\left(\cup_{n=1}^{\infty} S_n\right) \leq \sum_{n=1}^{\infty} M(S_n)$$

for any sequence $S_1, S_2, \ldots \in \mathscr{G}_{\mathscr{X}}$, is called an *outer measure* on $\mathscr{X}$. A subset S of $\mathscr{X}$ is said to be measurable with respect to the outer measure M, or simple *M-measurable*, if for all $R \subset \mathscr{X}$ we have

$$M(R) = M(R \cap S) + M(R \cap S').$$

This condition is called the *Carathéodory splitting condition.*

Note. By Lemma II.2.A we see that μ^+ is an outer measure on $\mathscr{X}$. In the proof of Theorem II.2.3 we will see that μ^+ is a measure on $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$. We first need two lemmas.

Lemma II.2.1. If M is an outer measure on the power set $\mathscr{G}_{\mathscr{X}}$ of $\mathscr{X}$ then the class $\mathscr{A}_M$ of all M-measurable sets $S \in \mathscr{G}_{\mathscr{X}}$ is a Boolean σ algebra, and the outer measure M restricted to $\mathscr{A}_M$ is a measure.

Lemma II.2.2. Every set R in the Boolean σ algebra $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is μ^+ measurable. That is, $\mathscr{A}_\sigma \subset \mathscr{A}_{\mu^+}$.

Note. We now have the equipment to prove Theorem II.2.3.

Revised: 4/21/2019