3.4. Bounded Inverses

Note. In this section, we give a condition equivalent to the inverse of a bounded linear operator being itself bounded (and hence continuous).

Definition. A linear operator $T$ from one normed linear space to another is bounded below if there is a $k > 0$ such that for all unit vectors $x$ we have $\|Tx\| \geq k$.

Note 3.4.A. Notice that if $T$ is bounded below by $k$ and if $x \in X$ is any nonzero vector, then $\|T(x/\|x\|)\| \geq k$ and so $\|Tx\| \geq k\|x\|$.

Note. Of course, to discuss $T^{-1}$, $T$ must be injective (one to one).

Theorem 3.6. Given an injective $T \in \mathcal{B}(X,Y)$ for which both $X$ and $Y$ are Banach spaces, the following are equivalent:

(i) $T^{-1}$ is bounded;

(ii) $T$ is bounded below;

(iii) $R(T)$ (the range of $T$) is closed.
Theorem. Bounded Inverse Theorem.

Given bijective $T \in \mathcal{B}(X,Y)$ where $X$ and $Y$ are both Banach spaces, $T^{-1}$ is bounded.

Note. The Bounded Inverse Theorem follows from Theorem 3.6 with $R(T)$ replaced with $Y$. 

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